Dynamics of a Rarefied Plasma in a Magnetic Field

By R. S. Sagdeyev, B. B. Kadomtsev, L. I. Rudakov and A. A. Vedyonov

The nature of the motion and properties of a high temperature plasma in a magnetic field is of particular interest for the problem of producing controlled thermonuclear reactions. The most general theoretical approach to such problems lies in the description of the plasma by the Boltzmann and Maxwell equations that connect the self-consistent electric and magnetic fields with the ion and electron distribution functions.

If the mean free path of the particles is appreciably larger than the characteristic dimensions of the system and if the time between collisions is greater than the appropriate characteristic time, then the collision term can be neglected in the kinetic equation or considered to be a small correction.

In the absence of collisions the plasma particles interact through long-range electromagnetic forces and the particle trajectories can be described by an equation of motion with self-consistent electric and magnetic fields. This is expressed mathematically by the fact that in the absence of collisions the solution to the kinetic equation

\[ \frac{df}{dt} + \mathbf{v} \cdot \mathbf{v} f + (e/M)(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} f = 0 \]  

(1)

can be represented by an arbitrary function of the first integrals of the characteristic equations for the motion of the individual particles

\[ dt = d\mathbf{x}/(eM)(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}). \]

The exact equations for the motion of a plasma in an electromagnetic field can only be solved in certain simple cases especially because the fields are influenced by the collective motion of all the particles. For a certain class of problems it is possible to work out a procedure for decreasing the number of variables and thus simplify the characteristic equations.

The physical system can be characterized by "fast" and "slow" variables. If we are investigating changes in the system over a long period of time, we can eliminate the fast variables and describe the state of the system in terms of the slow variables. The method of eliminating the fast variables from the kinetic equation was developed by Beliaev.*

We shall eliminate the "fast" variables for the particular case when the electric and magnetic fields change slowly in time and space; i.e., the conditions \( R_B/L \ll 1 \) and \( 1/\omega T \ll 1 \), where \( R_B \) is the Larmor radius, \( \omega \) is the Larmor particle frequency and \( L, T \) are the characteristic size and characteristic time of interest. The net particle motion consists of the "fast" Larmor rotation and the "slow" drift of the guiding centers.

The procedure for excluding the Larmor phase; i.e., the "fast" variable, from the equation of motion by means of a transformation to new "drift" variables \((v_{i\perp}, v_i, \mathbf{R}, \alpha)\) was worked out by Bogoliubov and Zubarev. Omitting the algebra, we shall present the final result, limiting ourselves to the zero and first order approximations in \( R_B/L \) and \( 1/\omega T \):

\[ dv_{i\perp}/dt = (v_{i\perp}/2\omega)d\omega/dt \]

(2)

\[ dv_i/dt = - e_0 \times (- e/M)\mathbf{E} + (v_{i\perp}^2/2\omega)\mathbf{v}_i + d\mathbf{w}/dt + v_i d\mathbf{e}_0/dt \]

(3)

\[ d\mathbf{R}/dt = e_\perp v_{i\perp} + (1/\omega) e_0 \times (- e/M)\mathbf{E} + (v_{i\perp}^2/2\omega)\mathbf{v}_i + d\mathbf{w}/dt + v_i d\mathbf{e}_0/dt, \]

(4)

where

\[ \frac{d}{dt} \equiv \partial/\partial t + v_i (e_\perp \mathbf{v}) + (\mathbf{w} \cdot \mathbf{v}). \]

(5)

Since the characteristic equations are known, we can easily write the kinetic equation for the distribution function \( f(v_i, v_{i\perp}, \mathbf{R}, \alpha, t) \) in terms of the drift variables:

\[ \frac{df}{dt} + (d\mathbf{R}/dt) \cdot \mathbf{v} f + (dv_{i\perp}/dt)\mathbf{v}_i f + (dv_i/dt)\mathbf{v}_i f + (d\alpha/dt)\mathbf{v}_i f = 0. \]

(6)

By expanding the resulting kinetic equation in powers of \( \omega^{-1} \), or more precisely, in powers of \( R_B/L \) and \( 1/\omega T \); we see that \( f_0 \) in \( f = f_0 + f_1 + ... \), does not depend on \( \alpha \) and one obtains an equation which contains only the "slow" variables:

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If rare collisions are taken into account, a small term on the right-hand side of the kinetic equation appears.

The electromagnetic fields and the distribution functions are connected through Maxwell’s equations. It is easy to show that to first order in $\omega^{-1}$ the velocity of the centre of gravity of an elementary volume of plasma, $n\vec{v} = \int (\vec{v} \times (v_z^2/2\omega)) \, e_0 + d\vec{R}/dt \int v_z \, dv_z \, dx$, does not coincide with the mean drift velocity, $n\vec{v} = \int (d\vec{R}/dt) f_0 v_z \, dv_z \, dx$, and is given by

$$n\vec{v} = \int (-\nabla \times (v_z^2/2\omega)) \, e_0 + d\vec{R}/dt \int v_z \, dv_z \, dx.$$  

With this expression, Maxwell’s equations can be written

$$\vec{V} \cdot \vec{E} = 4\pi e \int (f_0 - f_0^*) v_z \, dv_z \, dx, \quad (8)$$

$$\vec{V} \cdot \vec{B} = 0, \quad (9)$$

$$\vec{V} \times \vec{E} = -(1/c)\partial\vec{B}/\partial t, \quad (10)$$

$$\vec{V} \times \vec{B} = (4\pi e/c) \int [-\nabla \times (v_z^2/2\omega)] \, e_0 + d\vec{R}/dt \int (f_0 - f_0^*) v_z \, dv_z \, dx, \quad (11)$$

It may be seen from this equation that the plasma possesses diamagnetic properties. The set of Eqs. (6)–(11) completely describes the plasma when collisions are neglected. However, it should be stressed that this system of equations follow from the first non-vanishing approximation to the exact system of plasma equations when this exact system is expanded in terms of a small parameter, $\omega^{-1}$. This is the so-called drift approximation.

EQUATIONS FOR THE MACROSCOPIC MOTION OF THE PLASMA

Strictly speaking, conventional hydrodynamic considerations are valid when the plasma has a high density, low temperature and a corresponding collision mean free path which is much less than the characteristic dimensions of the plasma volume. However, Chew, Goldberger and Low and also Watson and Brueckner have shown that it is possible to obtain a closed system of equations for the first few moments of the distribution function. These equations are formally analogous to the usual system of magnetohydrodynamic equations with a non-isotopic pressure tensor when collisions are neglected. These authors obtained the moment equations by expanding the kinetic equation in powers of $\omega^{-1} = \epsilon H/Mc$ or in powers of the mass to charge ratio, $M/e$.

The similarity in the results obtained for these two apparently opposite cases is explained by the fact that the magnetic field deflects the particles and symmetrizes their velocity distribution in the plane perpendicular to the magnetic field. In this sense, the effect of the magnetic field is analogous to the effect of collision.

The gas-dynamic equations for the ion and electron gases can also be deduced from the kinetic equation in the drift approximation, as will be shown below.

The moments of the distribution function are defined by the following integrals

$$n = B \int J^{d2v}, \quad (12)$$

$$nu = B \int J^{d2v}, \quad (13)$$

$$\vec{p} = MB \int \langle v_z^2/2B \rangle J^{d2v}, \quad (14)$$

$$Q = MB \int \langle v_z - u \rangle J^{d2v},$$

where $d2v = d\mu dv$. This is the true volume element in velocity space.

If we assume that there is no heat flux $Q$ along the magnetic field then we can multiply the equation for the drift motion (4) and the kinetic equation (5) by various powers of $n_i$ and $v_i$ and integrate over the velocities. This procedure leads to a closed system of equations which describe the motion of the ionic component of the plasma

$$\partial n /\partial t + \vec{V} \cdot n (\vec{e}_0 \mu + \vec{u}) = 0 \quad (12)$$

$$Mdn_i/\partial t = -(\mu_i/M) \vec{V} \cdot \vec{B} - \partial \mu_i/\partial t \quad (13)$$

$$\partial \vec{e}_0 /\partial t = \vec{e}_0 \cdot \vec{e}_0 \vec{V} \cdot \vec{B} - \vec{e}_0 \vec{B}/\mu_i, \quad (14)$$

$$\partial \vec{e}_0 /\partial t = \vec{e}_0 \cdot \vec{e}_0 \vec{V} \cdot \vec{B} - \vec{e}_0 \vec{B}/\mu_i, \quad (15)$$

The relations which govern the pressure variation are given by Eqs. (14) and (15), which correspond to the adiabatic law with $\gamma = 3$ and 2 for the longitudinal and transverse motions, respectively. The transverse adiabatic invariant is the ion (or electron) magnetic moment. Maxwell’s equations (8)–(11) can also be rewritten by introducing the moments of the distribution function

$$\vec{V} \cdot \vec{E} = 4\pi e (n_i - n_e) \quad (8a)$$

$$\vec{V} \cdot \vec{B} = 0 \quad (9a)$$

$$\vec{V} \times \vec{E} = -(1/c)\partial\vec{B}/\partial t \quad (10a)$$

$$\vec{V} \times \vec{B} = (4\pi e/c) \{ - \vec{e} \vec{V} \times (\mu_i + \mu_e)/\vec{e} + \mu_i \vec{V} \cdot n (\vec{e}_0/\vec{e}) \} \quad (11a)$$

In order to describe the motion along the lines of force of the magnetic field by means of these equations, a special condition must be satisfied; namely, there must be no heat flux along the magnetic field.
lines of force. If this condition is violated then the equation \( d(pB^2/\rho^2)/dt = 0 \) is not valid, and the whole system of equations of motion (12)-(16) are inapplicable. In these cases a rigorous treatment has to be carried out on the basis of the more complicated system of Eqs. (7)-(11).

**HYDRODYNAMICS OF A LOW PRESSURE PLASMA**

The problem of determining the equilibrium state and stability of a magnetically confined plasma is perhaps one of the most interesting and important problems in plasma physics. If the magnetic field is very strong this problem can be attacked by using the magnetohydrodynamic equations with a non-isotropic pressure tensor. However, if the plasma is confined for a time greater than the collision time then the longitudinal and transverse pressures will be equalized. The pressure can then be regarded as isotropic in calculating the equilibrium state of the plasma. The anisotropy of the pressure should be taken into consideration only while investigating the stability for increments of time shorter than the collision time. However, it can be shown \(^4\) that the anisotropy of the perturbation pressure for an isotropic equilibrium pressure can only improve the conditions for stability. Therefore, when considering the stability of an isotropic plasma, one can use the ordinary hydrodynamic equations: the stability conditions obtained in this way are somewhat over-stringent.

We shall now consider the hydrodynamic description of a plasma when its pressure is appreciably lower than the magnetic pressure, i.e., \( \beta \equiv 8\pi p/B^2 \ll 1 \). In this case the magnetohydrodynamic equations can be simplified. Indeed, in such cases, the magnetohydrodynamic wave velocity \( c_A = (B^2/4\pi \rho)^{1/2} \) is much greater than the sound velocity \( c_s = (8\pi \rho)^{1/2} \). Therefore if we do not specifically consider the hydrodynamic waves we may take \( c_A \) to be infinite. This means that any perturbation of the plasma leading to a transverse displacement of the line of force is propagated instantaneously along the line of force and all the plasma along the line starts to move at once. The equation for this motion may be obtained by expanding the hydrodynamic equations in a power series in \( \beta \). Let us assume that \( \overline{p} = p_0 + e\overline{p}_1 + \ldots, \quad \rho = \rho_0 + e\rho_1 + \ldots, \quad \mathbf{v} = \mathbf{v}_0 + e\mathbf{v}_1 + \ldots, \quad B = C^{1/2} (B_0 + eB_1 + \ldots), \) substitute these expansions into the magnetohydrodynamic equations, compare coefficients of the same powers of \( e \) and then put \( e = 1 \). The following system of equations is obtained by this procedure:

\[
\begin{align*}
\mathbf{v} \times \mathbf{B}_0 \times \mathbf{B}_0 = 0, & \quad \mathbf{v} \cdot \mathbf{B}_0 = 0 \\
\rho \mathbf{v} \cdot \mathbf{v} = 0 & \quad \mathbf{v} = \mathbf{v}_0 + e\mathbf{v}_1 + \ldots, \quad B = C^{1/2} (B_0 + eB_1 + \ldots), \) substitute these expansions into the magnetohydrodynamic equations, compare coefficients of the same powers of \( e \) and then put \( e = 1 \).
\end{align*}
\]

By retaining only the zero-order terms we eliminate the other equations and can drop the subscript zero. If we do not consider force-free fields we obtain from Eq. (17)

\[
\begin{align*}
\mathbf{v} \times \mathbf{B}_0 &= 0 \\
\mathbf{v} \cdot \mathbf{B}_0 &= 0; \quad \text{i.e.,} \quad B = \Sigma f_i \mathbf{v}_i f_i \\
\end{align*}
\]

where the \( f_i \) are the external currents and the \( \phi_i \) are the corresponding scalar potentials which, generally speaking, are not single valued.

For simplicity we shall suppose that the currents do not change with time and that the conductors are immobile. Then \( \partial \mathbf{B}_i / \partial t = 0 \) and from Eq. (18) we obtain

\[
\mathbf{v} = e_0 \mathbf{v}_i + e_0 \mathbf{v} \times \mathbf{\phi} / \mathbf{B},
\]

where \( \mathbf{v}_i \) is arbitrary an function of \( r, t \) and \( \mathbf{\phi} \) is a function of \( r, t \) which satisfies the condition \( e_0 \cdot \mathbf{\nabla} \mathbf{\phi} = 0 \). Inserting this into Eq. (19) we must consider Eq. (19) to be an equation defining \( B_i \). Furthermore, as is well known, the condition for the existence of a solution must be fulfilled: the left-hand side of Eq. (19) must be orthogonal to the solutions of the homogeneous equation conjugate to the equation \( L \mathbf{B}_i = (\mathbf{v} \times \mathbf{B}_i) \times \mathbf{B}_0 = 0 \). This condition leads to

\[
\rho \mathbf{v} \cdot \mathbf{v} = 0; \quad U = -\int \mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) \, dl / B,
\]

where the integral is taken along any line of force. Together with Eqs. (20) and (21), these equations describe the behaviour of a low pressure plasma in the zeroth approximation. It should be noted that this approximation appears to be inadequate if the magnetic lines of force are not closed but cover the whole surface. Indeed, in this case, \( \phi \) must be constant on the surface, whereas \( \mathbf{v} \) retains only two degrees of freedom.

If we put \( \mathbf{v} = 0 \) in Eqs. (22) and (23) then conditions for equilibrium which are obtained can be transformed into the form

\[
\mathbf{e}_0 \cdot \mathbf{v} \rho = 0, \quad \mathbf{v} \rho \times \mathbf{v} U = 0; \quad U = -\int \mathbf{v} \mathbf{B} \, dl / B,
\]

i.e., the pressure \( \rho \) must be constant on the surface \( U = \text{constant} \).

Equations (2) through (23) can be used for investigating stability. We shall put \( \mathbf{v} = \mathbf{v}_0 + \mathbf{v}', \rho = \rho_0 + \rho', \mathbf{v} = \mathbf{v}_0 + \mathbf{v}, \mathbf{v} = \mathbf{v}_0 + \mathbf{v}, \) and neglect higher order quantities, we obtain linearized equations which describe small deviations from equilibrium. It is possible to show that these equations can be obtained from the variational principle \( \delta \int L \, dl = 0 \), where \( L = T - V, \) \( T = 1/2 \int \rho_0 (\mathbf{v}_0 \mathbf{v}_0) / \mathbf{B} \) is the kinetic energy and \( V \) is the potential energy of the plasma. If we introduce curvilinear coordinates \( (\xi_1, \xi_2, \xi_3) \) so that the lines \( \xi_3 \) coincide with the (closed) lines of magnetic force, then \( V \) can be written as follows:
where $g_{ik}$ is a component of the metric tensor and $g = \det g_{ik}$. For plasma stability it is necessary and sufficient that the potential energy $V$ be positive definite; i.e., its minimum value be positive. Minimizing Eq. (25) with respect to $\eta$, we obtain

$$V_{\min} = -\frac{1}{2} \int B (g / g_{ab}) \frac{1}{2} (\mathbf{n}_{l} \cdot \nabla \mathbf{p}_{0}) (\mathbf{n}_{l} \cdot \nabla U) d\xi d\xi_{a}$$

where

$$\frac{1}{2} \int \mathbf{g}_{\phi} (\mathbf{\nabla} \cdot \eta) d\mathbf{r}.$$  

which shows that a necessary and sufficient condition for the stability of a low pressure plasma is

$$\mathbf{v}_{jn} \cdot \mathbf{\nabla} U + (\gamma \rho_{0} / U) (\mathbf{\nabla} U)^{2} < 0.$$  

This condition is a hydrodynamic analogue to the stability criterion obtained from the kinetic equation by Tserkovnikov. This condition is necessary and sufficient for plasma stability. We also consider a homogeneous plasma that is stable when the pressure is isotropic. This case can be analyzed with the nonisotropic magnetohydrodynamic equations, but a more exact analysis involves the use of the kinetic equations. Both calculations lead to qualitatively consistent criteria for stability but the quantitative results are somewhat different. The situation here is quite analogous to the equilibrium pressure is nonisotropic another kind of instability appears in the plasma. In order to investigate this type of instability in its pure form we shall consider a homogeneous plasma that is stable when the pressure is isotropic. This case can be analyzed with the nonisotropic magnetohydrodynamic equations, but a more exact analysis would involve the use of the kinetic equations. Both calculations lead to qualitatively consistent criteria for stability but the quantitative results are somewhat different. The situation here is quite analogous to the Langmuir plasma oscillation problem.

Let us first make the assumptions that we can neglect the particle collisions, that the wavelength of the perturbation is much larger than the mean ion Larmor radius and that the interval of time is much less than the Larmor frequency. Under these conditions, the drift approximation (Eqs. (4)-(11)) can be used.

The unperturbed distribution function is chosen to be of the form

$$f_{0}(v_{1}^{2}, v_{2}^{2}) = \left[ n_{e} M^{\gamma} / 2 T_{r}^{\pi} \sqrt{2 \pi / T_{r}} \right] \exp \left\{ - (M / 2)(v_{1}^{2} / T_{r} + v_{2}^{2} / T_{r}) \right\}.$$  

On linearizing the initial equations for a small perturbation and performing a Fourier coordinate transformation and a Laplace time transformation, it is easy to obtain the characteristic equation for $\phi = \rho(k_{r}, k_{z})$:

$$\frac{\phi}{k_{r}} = \left( \frac{T_{r}}{M} \right)^{{\frac{1}{2}}} \frac{\beta_{\perp} [(T_{r} / T_{r}) - 1] - (1 + \beta_{\perp} - \beta_{\parallel}) (k_{r} / k_{g})^{2} - 1}{(\pi / 2)^{\frac{1}{2}} \beta_{r} T_{r} / T_{g}}.$$  

where the $z$ axis is in the direction of the unperturbed field $B_{0}$ and $\beta_{\perp}$, $\beta_{r}$ are defined by

$$\beta_{\parallel} = 8 \pi n_{0} T_{r} / B_{0}^{2}$$

$$\beta_{r} = 8 \pi n_{0} T_{r} / B_{0}^{2}.$$  

The perturbations increase exponentially with time if

$$\beta_{\perp} [(T_{r} / T_{r}) - 1] > (1 + \beta_{\perp} - \beta_{\parallel}) (k_{r} / k_{g})^{2} + 1.$$  

In the limiting case $B_{0} \to 0$, the instability condition is $T_{r} > T_{\parallel}$ and for $k_{g} = 0$ the characteristic equation is

$$(\phi / k_{r})^{2} = \left( B_{0} / 4 \pi n_{0} M \right) (1 + \beta_{\perp} - \beta_{\parallel}).$$  

This corresponds to the Alfvén magnetohydrodynamic branch. When $\beta_{r} > 1 + \beta_{\perp}$, the plasma is unstable and, when $B_{0} \to 0$, this instability condition reduces to $T_{\parallel} > T_{r}$.

For a magnetic field with axial symmetry this condition has also been obtained by a slightly different method.²

² For a magnetic field with axial symmetry this condition has also been obtained by a slightly different method.

## INSTABILITY OF A PLASMA IN A MAGNETIC FIELD WITH AN ANISOTROPIC ION VELOCITY DISTRIBUTION

If the equilibrium pressure is nonisotropic another kind of instability appears in the plasma. In order to investigate this type of instability in its pure form we shall consider a homogeneous plasma that is stable when the pressure is isotropic. This case can be analyzed with the nonisotropic magnetohydrodynamic equations, but a more exact analysis would involve the use of the kinetic equations. Both calculations lead to qualitatively consistent criteria for stability but the quantitative results are somewhat different. The situation here is quite analogous to the Langmuir plasma oscillation problem.

Let us first make the assumptions that we can neglect the particle collisions, that the wavelength of the perturbation is much larger than the mean ion Larmor radius and that the interval of time is much less than the Larmor frequency. Under these conditions, the drift approximation (Eqs. (4)-(11)) can be used.

The unperturbed distribution function is chosen to be of the form

$$f_{0}(v_{1}^{2}, v_{2}^{2}) = \left[ n_{e} M^{\gamma} / 2 T_{r}^{\pi} \sqrt{2 \pi / T_{r}} \right] \exp \left\{ - (M / 2)(v_{1}^{2} / T_{r} + v_{2}^{2} / T_{r}) \right\}.$$  

We shall now discuss the physical interpretation of the instabilities which arise from perturbations of a plasma with a nonisotropic temperature. Consider a density perturbation in an otherwise homogeneous plasma. At points where the density is high the magnetic field is decreased because of the diamagnetism of the anisotropic plasma. There also appears a gradient in the magnetic field and a curvature of the magnetic field lines as a result of the perturbation. In the limiting case $B_{0} \to 0$, the instability condition is $T_{r} > T_{\parallel}$ and for $k_{g} = 0$ the characteristic equation is

$$(\phi / k_{r})^{2} = \left( B_{0} / 4 \pi n_{0} M \right) (1 + \beta_{\perp} - \beta_{\parallel}).$$  

This corresponds to the Alfvén magnetohydrodynamic branch. When $\beta_{r} > 1 + \beta_{\perp}$, the plasma is unstable and, when $B_{0} \to 0$, this instability condition reduces to $T_{\parallel} > T_{r}$.

² For a magnetic field with axial symmetry this condition has also been obtained by a slightly different method.
see that these forces tend to increase the amplitude of the initial perturbation.

All the calculations described heretofore are based on linearized equations and, naturally, they give no information about the limiting amplitude of the undamped oscillations of an anisotropic plasma.

We have carried out the analysis to second order approximation for a number of simple limiting cases and have shown that the appearance of the instability described above leads to a transfer of kinetic energy from the transverse to the longitudinal motion when \(T_1 > T_\|\) and in opposite direction if \(T_1 < T_\|\). Therefore, it is reasonable to assume that the instability tends to develop until the longitudinal and transverse energies are equalized; i.e., until the transverse and longitudinal temperatures are approximately equal.

**STABILITY OF A PINCHED CYLINDRICAL PLASMA USING THE KINETIC EQUATION**

In this section the problem of the stability of a high-temperature, pinched, cylindrical plasma is treated. We shall assume that all the conditions for the validity of the drift approximation are satisfied. For simplicity we shall neglect the electron temperature in comparison to the ion temperature. With this assumption, we can use the starting equations of the previous section.

We shall consider that in the equilibrium state, the plasma cylinder is uniform along the \(z\) axis and along the azimuth \(\phi\). Outside the cylinder \(r = a\) the non-zero components of the field are \(H_{\phi}(r) = 2frc, H_{z} = 0\). Inside the cylinder, we have \(B_{\phi}(b) = 0\) and \(B_{z}(b) = B_\| = \text{constant}\). The indices \(i, e\) relate to the fields internal and external to the cylinder, respectively. The ion distribution function \(f_i\) is defined at the beginning of the previous section. We neglect the electric field \(E_\|\), assuming that it is eliminated by the motion of the electrons along the \(z\) axis. This is true when the condition \(\partial\phi/\partial t \ll \omega_{pe}\) is satisfied, where \(\omega_{pe}\) is the Langmuir electron frequency.

On the surface of the plasma cylinder the following condition should be satisfied:

\[
8\pi\beta_\perp = H_{\phi}^2 + H_z^2 - (B_\perp)^2,
\]

where

\[
\beta_\perp = \int (Mv_i^2/2) (i_\phi + i_t) d\mathbf{v}.
\]

The motion of the boundary obeys the drift equation \(\mathbf{u}_\perp = (\mathbf{E} \times \mathbf{B})/B_\perp^2\). The perturbed quantities are taken to be of the form \(F(x, \phi, z, t) \sim F(x, \phi, z, 0) \exp[i(kz + m\phi + \omega t)]\). One then obtains the following dispersion relation in the usual way:

\[
\alpha^2 = \frac{k^2(1 + \beta_\perp - \beta_\parallel)}{1 + \beta_\perp + (\pi k/B_\parallel)^2} (\omega/c_\Lambda)^2 - \left(\frac{\omega/c_\Lambda}{\omega_0 + k\epsilon} \frac{\partial f_i}{\partial v_\|}\right)^2,
\]

where

\[
c_\Lambda = \frac{H_\parallel}{4\pi n_0 M} \gamma^2,
\]

and

\[
h_i = B_\parallel H_{\phi}(r)(a),
\]

\[
h_\parallel = H_\perp(0) H_{\phi}(0)(a).
\]

The form of this equation has much in common with the dispersion equation obtained in the magnetohydrodynamic approximation by Shafranov.

We shall now consider a number of particular cases.

1. \(T_1 = T_\parallel\): In this case the instability criterion coincides precisely with the criterion obtained by Shafranov for \(\gamma = 2\). However, the dependence of the perturbed quantities on the wave vector will differ from the dependence obtained from the hydrodynamic approximation.

2. \(I = 0, H_{\phi}(0) = 0, T_1 \neq T_\parallel\): The magnetohydrodynamic treatment did not show any instability under these conditions. In our treatment, however, it is possible to have an instability in a plasma cylinder confined by a longitudinal magnetic field when there is no axial current. The instability criterion can be written explicitly when the perturbations along the axis are greater than the radius of the magnetically confined plasma. The plasma column will be unstable if

\[
H_\perp^2 + B_\parallel^2 + 4\pi n_0 (T_1 - T_\parallel) < 0.
\]

This condition can be fulfilled only if \(T_1 < T_\parallel\).

In the limiting case of short wavelength perturbations, an instability may occur with \(T_1 > T_\parallel\). The criterion for the existence of such an instability is given by

\[
|\alpha(\omega = 0)|^2 < 0.
\]

3. The vibrational branch: Because of the thermal motion there is a damping of magnetosonic waves in the cylinder analogous to Landau damping. For small \(k\), when the damping is small, the imaginary part of \(\omega\) is

\[
\omega = \frac{8\pi n_0 T}{B_\parallel^2} \frac{H_\perp^2}{B_\parallel^2} \times \frac{(\omega c_\Lambda)^2 k(\pi M/2T)^{3/2} \exp{(M/2T)(\omega/k)^2}}{2(m + k)(2(m + 1))}.
\]

The instability associated with temperature anisotropies could be observed in experiments on so-called adiabatic heating when only the transverse temperature decreases with decreasing magnetic field and the longitudinal temperature remains constant. This situation might arise at high temperatures where collisions can be neglected.

In experiments on the fast compression of a plasma column the compression time can be less than the mean time between ion collisions. This automatically leads to a temperature anisotropy that can make the condition for stability more rigid.
NON-LINEAR ONE-DIMENSIONAL MOTION OF A RAREFIED PLASMA

We shall now investigate the motion of a rarefied plasma far from an equilibrium state. We shall restrict ourselves to the case when all the variables will depend only on one space coordinate $x$ perpendicular to a magnetic field directed along the $z$ axis $B = (0, 0, B(x))$. The system of hydrodynamic equations (12)–(16) are valid subject to the conditions that the space and time gradients are small, i.e., the Larmor radius and period are small compared to the appropriate linear dimensions and time. However, generally speaking, these conditions for the arbitrary motion of a plasma will not be fulfilled after a certain time.

Indeed, the formal similarity between the system of Eqs (12)–(16) and the ordinary gas-dynamic equations with $\gamma = 2$ enables us in the case of a rarefied plasma in a magnetic field to immediately generalize Riemann’s solution corresponding to the formation of a shock wave. Steep velocity and temperature gradients in an ordinary shock front lead to a considerable increase of the entropy due to viscosity and heat conduction. However, in a rarefied plasma the formation of discontinuities ought to take place in a different manner since collisions are of no importance.

A hypothesis concerning the possible existence of a shock wave was advanced in an earlier paper where the non-conservation of the ion magnetic moment in a field with large gradients was considered as a mechanism leading to non-adiabaticity.

The rigorous treatment of this problem is very complicated since the motion even of a single charged particle in an arbitrary nonuniform field cannot be determined without the aid of a computer.

The simplest idealization of the processes inside the wavefront would be to consider the change in the Larmor energy across a magnetic discontinuity. It appears, however, that the non-adiabatic increase of the Larmor energy of the individual ions does not imply that the random energy is also non-adiabatically increased since there is a grouping of the Larmor phases of the ions.

This grouping of phases leads to a local oscillation of the ions. In a self-consistent problem this might also give rise to oscillations. Damping can in principle take place in the absence of collisions, e.g., due to the Landau mechanism or on account of phase mixing in a nonuniform magnetic field, and the damping length will determine the width of the shock wave. With the curvature of the velocity profile increasing, the condition $T \gg \omega^{-1}$ is violated before the condition $L \gg R_B$ since $T \sim \lambda \left[(B^2/4\pi p) + u^2 \right]^{\gamma/2}$. Therefore, the wavelength $\lambda$ of the oscillation may be expected to be of the order $\lambda \sim \left[(B^2/4\pi p) + u^2 \right]^{\gamma/2}$. For a low pressure plasma this becomes $\lambda \sim B/[(4\pi p)^{3/2}]$, $\omega_p = \sqrt{4\pi n e^2/M}$, where $\omega_p^2 = 4\pi ne^2/M$. However, it may be shown by using the exact equations for the ion and electron motion that for vanishing plasma pressure, the assumption of plasma quasi-neutrality and electron adiabaticity is sufficient to ensure the applicability of Riemann’s solution, instead of the condition $\omega T \gg 1$ (the condition $L \gg R_B$ is then fulfilled identically since $\lambda \sim 0$).

In ordinary gas dynamics the only type of stationary motion is a stationary shock wave. In a rarefied plasma there exists still another solution which is a travelling magnetosonic wave with a finite amplitude.

If the thermal motion is neglected, the equation connecting the wave velocity with the amplitude of the magnetic field has the form $u^2 = (B_1 + B_0)^2/16\pi p_0$, where the field amplitude $B_1$ of the wave cannot exceed $3B_0$.

The correction to the velocity due to thermal motion is

$$
u_1 = \frac{\alpha^2 (u_0^2 - \gamma)^{\gamma/2} - 3u_0^2}{2u_0^2 - \gamma - 2u_0^2 (u_0^2 - \gamma)^{\gamma/2}},$$

where $u_0 = B_0/(4\pi p_0)^{1/2}$, $\gamma = (B_1 + B_0)/2\pi p_0$, $\alpha^2 = 2T_0/M$.

This equation shows that the transition to small amplitude waves leads to absurd results since the denominator vanishes. This fact is probably associated with the existence of a damping in the linear approximation analogous to Landau damping.

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* Ed note In another version of this paper the term $3u_0/4$ is written $3u_0/2$