The Stability of a Constricted Gas Discharge

By R. J. Tayler *

One of the most obvious methods of attempting to confine a hot plasma is to use the pinch effect of the self magnetic field of an axial current. This is illustrated in Fig. 1. A cylindrical system is most easily studied theoretically, although it is probably necessary to close the system into a torus, or similar shape, to avoid heat and particle losses at electrodes. Although some consideration has been given to toroidal geometry, most complete results have been obtained for infinite cylindrical systems and only these will be discussed here. This paper is to be regarded as a progress report on an unfinished problem and an indication is given of aspects of the problem which, it is hoped, will be studied soon.

The original idea of using the pinch effect was that of constricting a gas discharge to an extremely small radius when the only heat losses from it would be radiation losses, principally bremsstrahlung. It was hoped that power would eventually be achieved by passing a high enough current to reach thermonuclear temperatures, and by containing the system for a long enough time. At an early stage, it became clear that this simple idea must be modified because of the presence of instabilities (see Fig. 2). These were observed and predicted at about the same time. When attempts were made to understand the instabilities and to predict methods of stabilisation, it was found that a high degree of constriction did not appear to be possible if the discharge were to be stable. Thus, study is now made of rather fat discharges in a large tube.

It should be noted that a linear stability theory, such as was considered in Refs. 6 and 7, does not immediately prove that a highly constricted discharge cannot be used. Although such a discharge is unstable, non-linear stabilising forces may prevent it from approaching the walls of its containing vessel. At the same time, the mass motions of the discharge may lead to additional heating, which would be an advantage at high temperatures when Joule heating becomes relatively inefficient. The non-linear problem has not received adequate investigation to date. What evidence there is suggests that the non-linear forces are not very effective unless the plasma is fairly close to stability. Observations are, unfortunately, very easy only for completely unstable discharges. Thus, the early work at Harwell was done in Pyrex tubes and the entire current channel could be photographed. Such a photograph is shown in Fig. 3. Generally the most developed instability appeared to be a helix of pitch 45°; this result is remarkably independent of the particular apparatus used. A very detailed analysis of instabilities has been made by Allen, using a racetrack torus. When stabilisation is seriously considered, the tube has a conducting wall and visual observations can be made only through slits. In this case, the presence of the slits may effect stability locally. Probe and microwave measurements are difficult to interpret, unless their time resolution is very good, and visual measurements may be ambiguous because the most frequent occurrence is for the light to be given out by impurities, which may not occupy the same volume as the main current channel.

The full problem of discharge stability, even for a cylindrical system, is extremely complicated. In particular, a discharge will not be in simple hydrostatic equilibrium and, even in the absence of instabilities, the finite electrical conductivity will cause gradual changes in the magnetic field configurations. In most of the work described in this paper, it is assumed that steady motions and field penetration occur so slowly that, in the stability problem, the discharge can be supposed ideally conducting and at rest. Two theoretical models are considered here. The first approach is extremely naive: plasma dynamics is completely neglected and only the forces exerted by the electromagnetic field are considered. The second approach is to use Maxwell's equations combined with a simple one-fluid hydrodynamics—classical hydromagnetics. A more sophisticated procedure would be to start from Boltzmann's equation for the several types of particle present and to try to derive a hydrodynamic system. As an example of this procedure there is the work of Watson and co-workers and of Chew, Goldberger and Low. In the simplest problems that have been considered, essentially similar results have been obtained from the Kinetic Theory, Hydromagnetic and even the Magnetostatic approach.

The remainder of the paper is arranged as follows. In the next section the magnetostatic or extensible

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wire model of the discharge is studied. The following two sections are based on the hydromagnetic equations: in one, the gradual loss of stability of the stabilised pinch, due to field diffusion, is discussed, and, in the other, another type of stable discharge configuration is introduced. In the final section there is a discussion of further problems which must be studied. Gaussian units are used throughout for the electrical quantities.

THE WREGLING DISCHARGE

The first model of the discharge considered is one in which all the plasma dynamics is neglected and only electromagnetic forces are considered. Thus the discharge is replaced by an extensible wire carrying a current. When this model was introduced, interest was concentrated on the highly constricted discharge and the discharge was treated as a very thin wire. Given perturbed configurations of the wire were considered and the forces acting on the configuration were calculated in a magnetostatic approximation. First calculations were made for the outward, instability-producing forces on a free discharge; later calculations were made of restoring forces due to conducting walls. The results obtained from this model are much better than might be expected and for the linear terms the stability criteria are in good agreement with those found for a hydromagnetic model. In addition it is quite easy to calculate non-linear terms in the forces and to obtain an estimate of the amplitude to which given perturbations can grow before the net force is inward. This is a somewhat academic calculation because as soon as non-linear terms are considered configuration-changing forces are introduced and all the modes of instability interact. However, it does give a first clue to the importance of non-linear forces and suggests that they cannot be very effective unless the plasma is fairly close to stability; that implies that it is not highly constricted.

The results for one particularly simple configuration are as follows. Suppose a discharge of radius \( r_0 \) is in a tube of radius \( \Delta r_0 \) and carries a current \( J \). The perturbed discharge takes a helical form given by the equations

\[
x = A \cos kz, \quad y = A \sin kz.
\]

The outward (self) force and inward (image) force are expanded for small values of \( Ak \) and \( 1/\Lambda \) and take the form:

\[
\frac{r_0 c^2 F_{\text{self}}}{J^2} = A k X_0 \left[ - \ln X_0 + \ln 2 - \gamma + \frac{1}{2} \right] - A^3 k^3 X_0 \left[ - \ln X_0 - \ln 2 - \gamma + \frac{5}{6} \right] + A^4 k^4 X_0 \left[ - \ln X_0 + 4 \ln 2 - \frac{243}{64} \ln 3 - \gamma + \frac{23}{32} \right]
\]

and

\[
\frac{r_0 c^2 F_{\text{image}}}{J^2} = - A k X_0 \left[ K_0 (2AX_0) + K_2 (2AX_0) \right] \left[ I_0 (2AX_0) + I_2 (2AX_0) \right] - 2 A^3 k^3 X_0 \left[ K_1 (2AX_0) + K_3 (2AX_0) \right] \left[ I_1 (2AX_0) + I_3 (2AX_0) \right] + A^4 k^4 X_0 \left[ K_0 (3AX_0) + K_2 (3AX_0) \right] \left[ I_0 (3AX_0) + I_2 (3AX_0) \right] + \frac{243}{64} \left[ K_1 (3AX_0) + K_3 (3AX_0) \right] \left[ I_1 (3AX_0) + I_3 (3AX_0) \right].
\]
where $F_r$ is the radial force, $\gamma$ is Euler's constant $0.5772\ldots$, $I_n$ and $K_n$ are modified Bessel functions of the first and second kinds and $X_\theta$ is a non-dimensional wave number $kr_0$. The correcting terms to (2.2) and (2.3) are of order $(Ak)^2$ and $1/\Lambda^3$.

Results have been calculated from (2.1) and (2.2) for several values of $\Lambda$ and they are shown for one value of $\Lambda$ ($\Lambda = 20$) in Fig. 4. It can be seen from this figure that the change in wave number, $X_\theta$, between a perturbation which is completely stable and one for which the outward force vanishes when it is halfway across the tube, is very small. Values of wavelengths and amplitudes, for which the net force vanishes, are shown in Table 1 where $R_0 = \Lambda r_0$ is the tube radius.

Table 1. Critical Values of Amplitudes for Which Outward Force Vanishes

<table>
<thead>
<tr>
<th>$1/\Lambda X_\theta$</th>
<th>1.49</th>
<th>1.46</th>
<th>1.39</th>
<th>1.29</th>
<th>1.14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A/R_0$</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Fuller details of this work are given in Ref. 10.

It would also be possible to include the effect of an axial magnetic field inside the discharge channel, by introducing a tension into the wire; however, this problem has been studied only on the hydromagnetic model. It is possible to show that an external magnetic field cannot completely stabilise helical perturbations of the type considered here.

In more recent work, by Curtis and Roberts, an attempt has been made to follow the time development of the wriggling discharge by using an electronic computer. The model of this section is further simplified, by replacing the exact static forces by an approximation better suited to numerical work. The equation of motion

$$\rho \ddot{\mathbf{x}} = \mathbf{F}_{\text{self}} + \mathbf{F}_{\text{image}}$$

is then solved.

The parameters taken were: tube radius, $R_0$; length of discharge, $10 R_0$; and number of points along discharge channel, either 100 or 300. The discharge radius was not included as an explicit parameter, as the logarithmic term in the self force was replaced by a constant; this is roughly equivalent to saying that the correct value of the force is used for some mean wavelength of instability. The value of the logarithm taken corresponded to a discharge fatter than the one considered in the static problem above ($\Lambda \approx 10$).

Because of the finite number of points along the discharge, there is a cut-off in possible perturbation wavelengths. Some smoothing had also to be introduced into the integration procedure, because of the propagation of rounding-off errors, and this introduced a further wavelength cut-off. This cut-off might be considered in some sense equivalent to the introduction of an axial magnetic field which stabilises short wavelengths. With these conditions the discharge moved across about 50% of the tube radius. Although the fatter discharge should be more stable than the results given in Fig. 4, it appears to be even more stable than would be expected. The full interpretation of these results has not yet been made.

### THE STABILISED PINCH

In this section the stability problem is studied, with the hydromagnetic equations as starting point. These are, for an ideally conducting fluid of pressure $p$, density $\rho$, and ratio of specific heats $\gamma$, carrying electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and current and charge densities $\mathbf{j}$ and $Q$,

$$\frac{\rho}{\gamma} \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{j \times \mathbf{B}}{c},$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v},$$

$$\frac{1}{\gamma} \frac{d\rho}{dt} = \nabla \cdot \rho \mathbf{v},$$

$$\mathbf{\nabla} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j},$$

$$\mathbf{\nabla} \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},$$

$$\mathbf{\nabla} \cdot \mathbf{B} = 0,$$

$$\mathbf{\nabla} \cdot \mathbf{E} = 4\pi Q$$

and

$$\mathbf{E} + \frac{\mathbf{\nabla} \times \mathbf{B}}{c} = 0.$$

It is supposed that an infinite cylinder of conducting fluid, of radius $r_0$, is surrounded by a vacuum containing a magnetic field $B_0(0, r_0/r, b_0)$ [in $(r, \theta, z)$ coordinates]. In turn, the vacuum is surrounded by a conducting wall at radius $\Lambda r_0$. The stability of a
Stability of a Constricted Discharge

System in which none of the equilibrium quantities depend explicitly on \( \theta \) and \( z \) is investigated by considering departures from equilibrium in which any variable \( q \) has the form

\[
q = q_0 + q_1(r, t) e^{i(m\theta + kz)}.
\]  
(3.9)

By solving the complete set of equations (3.1) to (3.8) for perturbations of the form (3.9), it has been shown by several authors that there exist stable configurations of this system, if the magnetic field inside the conducting fluid is constant and purely axial, having the value \((0, 0, B_0)\). In this case disturbances corresponding to \( m = 0 \) and \( m = 1 \) are the most difficult to stabilize but there are sets of values of \((b_0, b_1, \Lambda)\) for which complete stability is obtained. These results are illustrated in Fig. 5. Chandrasekhar, Kaufman and Watson have obtained the same stability criteria, for a plasma in which collisions are neglected and Boltzmann's equation is taken as the starting point.20

In fact, such discontinuous magnetic fields will not be realised in practice. Even if they could be obtained, the finite conductivity of the discharge would lead to a gradual interpenetration of the axial and azimuthal fields and to changes in the discharge radius. As it has also been shown that the discharge is unstable if the current density in the plasma is uniform and \( b_0 \) and \( b_1 \) are equal, it appears that at some stage in the interpenetration stability is lost. The field diffusion proceeds at a rate determined by a penetration time, \( \tau_p = 4\pi\alpha\sigma_0/\epsilon^2 \) where \( \alpha \) is the relevant electrical conductivity. If instabilities occur, a characteristic instability time \( \tau_1 \) is the time taken by a sound wave crossing the plasma. For high temperature plasmas \( \tau_p \gg \tau_1 \). Because of this inequality, it seems possible to divide the problem into two parts; \( a_1 \), the penetration problem, in which stability is assumed and the rate of field diffusion is calculated; and \( b_1 \), the stability problem, in which stability is investigated assuming that the plasma is static and ideally conducting.

Penetration

The penetration problem has not been studied in great detail. Neglecting the complications due to plasma motions and anisotropic electrical conductivity, a simple calculation has been made of the diffusion of an axial field out of a plasma and of an azimuthal field into it. With the added assumption that the axial magnetic flux and the azimuthal current remain constant, the equations for the time and space dependence of the plasma magnetic fields become:

\[
B_0/B_a = R + 2 \sum_{\nu=1}^{\infty} \frac{F_0(\alpha_0 R)}{\alpha_0 [F_0(\alpha_0)]^2} \exp - \alpha_0^2 T,
\]  
(3.10)

\[
B_0/B_a = b_{a0} + (b_{00} - b_{a0})/\Lambda^2 + 2(b_{00} - b_{a0}) \sum_{\nu=1}^{\infty} \frac{F_0(\beta_0 R)}{\beta_0^2 [F_0(\beta_0)]^2} \exp - \beta_0^2 T
\]  
(3.11)

where: \( F_0(\alpha_0) = 0; \)

\[
J_0(\beta_0) = - \frac{2}{(\Lambda^2 - 1)} \beta_0;
\]  
(3.12)

\[ T = t/\tau_p; \quad R = r/\epsilon_0; \quad b_{a0} \quad \text{and} \quad b_{a0} \quad \text{are the initial values of} \ b_0 \quad \text{and} \ b_1; \quad \text{and the} \ J_n \quad \text{are Bessel functions.}
\]

Results have been calculated for the special case of \( \Lambda = 2 \) and \( b_{a0} = 0 \) and these are shown in Figs. 6 and 7. The radial dependence of both axial and azimuthal fields is shown for several values of the time. It can be seen that quite considerable field penetration can occur in time \( 5 \times 10^{-3} \tau_p \) and, in fact, it will be seen that times less than this are of importance in stability problems. However, for a very hot plasma, even \( 10^{-3} \tau_p \) will be greater than \( \tau_1 \). Although an actual discharge does not have a constant scalar conductivity, the results of Figs. 6 and 7 are probably qualitatively correct in showing that a rather small fraction of the relevant penetration time suffices for noticeable field diffusion. The results can be generalised slightly; if \( \sigma \) is time- but not space-dependent, exactly the same equations (3.10), (3.11) hold, provided that a new time coordinate is introduced, defined by

\[
T' = \int_0^T \frac{\sigma_0}{\sigma(T')} \ dT',
\]  
(3.13)

where \( \sigma_0 \) is the initial value of the conductivity.

![Figure 5. Stability diagram for surface current. For several values of wall radius \( \Lambda \) the regions of stability for \( m = 0 \) and \( m = 1 \) are shown. Stability occurs in the region above the curve marked with the given values of \( \Lambda \) and below the curve \( 1 + b_0^2 = b_1^2 \). Complete stability for given \( \Lambda \) occurs when a point lies in the stability region for \( m = 0 \) and \( m = 1 \).](image-url)
Stability

The stability problem has been treated by an energy principle method due to Bernstein, Frieman, Kruskal, and Kulsrud. This has been developed from one first used by Lundquist; similar results have also been used by Hain, List and Schlüter. Bernstein et al. show that if a perturbation of a plasma is considered in which the element initially at \( r(0) \) moves to \( r(0) + \delta r \), the stability of the plasma against this perturbation can be determined by examining the sign of the change \( \delta W \) in the potential energy of the system. If \( \delta W \) is minimised with respect to all perturbations \( \delta r \) satisfying certain conditions on the plasma-vacuum interface, and the minimum \( \delta W \) is negative, then the system is unstable. Otherwise it is stable. The energy principle will not be described in detail here; the results for the pinched discharge, which will be briefly discussed, are given fully in Refs. 25 and 26.

It is supposed that in the central region of the plasma there is only an axial field \( B_\theta \), but that axial and azimuthal fields are mixed in the outer fraction \( \epsilon \) of the discharge radius. For mathematical simplicity, the field profiles taken are not those given by the solution of the penetration problem (Figs. 6 and 7). Instead it is supposed that both \( B_x \) and \( B_y/r \) are linear with radius in the current layer. Thus

\[
B = \{0, B_\theta x/r_\theta, B_\theta [b_0 + (b_0 - b_1)x]\},
\]

where

\[
r = r_\theta (1 - \epsilon) + r_\theta \epsilon x.
\]

For small \( \epsilon \), it can be seen from Figs. 6 and 7 that this approximation to the interpenetrating fields is probably quite good; in particular it appears justified to take the same \( \epsilon \) for both \( B_\theta \) and \( B_x \). The field and pressure profiles for one particular set of values of \( b_0, b_1, \lambda \) and \( \epsilon \) are shown in Fig. 8.
\[ \delta W = -B_0^2 \xi^2 \frac{(m + b_0 X_0)^2}{X_0} \]
\[ \times \left[ \frac{K_m(X_0)I_m(\lambda X_0) - I_m(X_0)K_m(\lambda X_0)}{K_m(X_0)I_m(\lambda X_0) - I_m(X_0)K_m(\lambda X_0)} \right] \]
\[ + B_0^2 \xi^2 \delta^2 X_0 [X_0(1 - \epsilon)] \]
\[ + B_0^2 \int_{r_0}^{r_*} \frac{r}{(r^2 - b_0^2)^3} \left( \frac{m B_0^2}{r^2} - \frac{2B_\theta}{r^2} \right) \] 
\[ + \frac{1}{m^2 + \lambda^2 b_0^2} \left( \frac{m B_0^2}{r} - \frac{2B_\theta}{r^2} \right) \]
\[ + 2r \xi \frac{\lambda b_0}{m^2 + \lambda^2 b_0^2} \left( \frac{m B_0^2}{r} - \frac{2B_\theta}{r^2} \right) \]
\[ + r^2 \xi \frac{1}{m^2 + \lambda^2 b_0^2} \left( \frac{m B_0^2}{r} + \frac{2B_\theta}{r^2} \right) \]  
\[ (3.16) \]

In equation (3.16), \( K_m, I_m \) are modified Bessel functions, \( \xi \) is the radial component of \( \xi \) and \( \xi_0 \) its values at the inner and outer boundaries of the current sheet, \( X_0 = b \), and the prime denotes differentiation with respect to \( r \) in the integral and with respect to the argument of the Bessel functions elsewhere. The first term in \( \delta W \) is the vacuum contribution, the second comes from the plasma centre and the third from the current sheet. If \( \epsilon \) tends to zero, it can be shown the third term approaches the value \(-1\). In this case, the condition that \( \delta W \) vanishes reduces to the stability criterion for the stabilised pinch given in Ref. 7.

The expression for \( \delta W \) is minimised for \( \xi \)'s of the form
\[ \xi = \xi_0 [1 + \epsilon \Sigma a_n x^n] \]  
(3.17)
so that
\[ \xi_1 = \xi_0 [1 + \epsilon \Sigma a_n] \]  
(3.18)

With the form (3.17) for \( \xi \) and (3.14) for the magnetic field, the integral in (3.16) can be evaluated either exactly or as a power series in \( \epsilon \).

One particular problem has first been considered. It is assumed that initially
\[ \epsilon = 0, \quad \lambda = 2, \quad b_0 = 0, \quad b_1 = 0.9 \]  
(3.19)
and that field diffusion occurs subject to the following three conditions:
(i) \( B_0 \) remains constant,
(ii) \( \lambda \) remains constant,
(iii) The total axial flux remains constant.

Perturbations have been considered with \( m = 0 \) and \( m = 1 \); these were the types of perturbations most difficult to stabilise with surface currents. At first, trial functions are considered with only one arbitrary parameter; afterwards the number of parameters is increased.

In the case of \( m = 0 \), it is easy to see that \( \delta W \) is monotonically increasing with \( X_0 \); thus we need only determine its sign at \( X_0 = 0 \). For \( m = 1 \), no such simple result holds. For \( m = 0 \), one-parameter trial functions have been considered for several values of \( n \) and \( \epsilon \). The resulting minimum values of \( \delta W \) are shown in Fig. 9. It can be seen that for each value of \( \epsilon \) considered, \( \delta W \) has its minimum value for \( n = 3 \). It is also easy to show algebraically that the minimum \( \delta W \), for one-parameter trial functions, approaches the asymptotic value shown as \( n \rightarrow \infty \).

It is possible to find the value of \( \epsilon \) for which the minimum point on the curve, corresponding to those shown in Fig. 9, lies at \( \delta W = 0 \). This is the critical value at which stability ceases in the one-parameter approximation. Corresponding critical values of \( \epsilon \) have been found for trial functions with up to four parameters and these are shown in Table 2. It can be seen that the value of \( \epsilon \) changes only very slightly in going from one to four parameters. These results suggest that the critical value of \( \epsilon \) converges well as more parameters are added, and the trial functions \( \xi \) also appear to be converging.

### Table 2. Values of \( \epsilon \) for Which Stability Is Lost for One-, Two-, Three- and Four-Parameter Trial Functions; \( m = 0 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \epsilon_{\text{crit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.157</td>
</tr>
<tr>
<td>2</td>
<td>0.156</td>
</tr>
<tr>
<td>3</td>
<td>0.150</td>
</tr>
</tbody>
</table>

In the case of \( m = 1 \) it is more difficult to obtain complete results because the value of \( X_0 \) which gives the minimum \( \delta W \) is unknown. Results must therefore be obtained for several values of \( X_0 \) and the minimum value deduced. For one-parameter trial functions, it is found that \( \delta W \) has its minimum value for \( n = 1 \), and for such trial functions the critical value of \( \epsilon \) at which stability ceases has been found. Corresponding values of \( \epsilon \) have also been found for two- and three-parameter trial functions. These results are shown in Table 3; they are not as accurate as those in Table 2 because of the amount of interpolation involved. However, the critical values for the two- and three-parameter cases are very close, both being between 0.185 and 0.190. It is clear that, in this case, the one-parameter trial function does not give an accurate idea of the critical value of \( \epsilon \); it is impossible to judge whether closeness of the two- and three-parameter values is a sign of convergence, without also doing calculations for \( n = 4 \) and above. It seems likely, for this special problem, that the \( \epsilon \) at which stability is lost is close to 0.15 and that the axisymmetric modes are the least stable.

In addition to this special problem, qualitative stability criteria for general field configurations have been obtained, using only one-parameter trial functions. For the case of \( m = 0 \), calculations have been
made for $\varepsilon = 0.1$ and $0.2$ and for several values of $\Lambda$. A one-parameter trial function with $n = 3$ was taken and, to this approximation, stability curves were drawn in the $b_1, b_0$ plane. These correspond to the curves for $\varepsilon = 0$ in Fig. 5 and they are themselves shown in Figs. 10 and 11. The upper curve in these figures is obtained from the condition that, for given $b_0$, there is a maximum value of $b_1$ for which a plasma can be contained. The equation of this curve is

$$1 + b_0^2 + \frac{3}{2} \varepsilon - \frac{3}{2} \varepsilon^2 = b_1^2.$$  \hspace{1cm} (3.20)

These stability curves should probably be a good approximation to the actual stability curves; in any case they always overestimate the region of stability.

Finally some calculations have been made with one-parameter trial functions with $m = 1$ and $\Lambda = 2$. The results of Table 3 indicate that stability curves obtained from these results cannot be more than qualitatively correct. Nevertheless, we have illustrated in Fig. 12 how, on the one-parameter approximation, the region of stability for both $m = 0$ and $m = 1$ shrinks as $\varepsilon$ increases.

Although the results obtained above indicate that stability is lost for quite a small value of $\varepsilon$, the temptation to consider only the terms in the energy that are linear in $\varepsilon$ must be resisted. For the special problem, the critical value of $\varepsilon$ for axisymmetric perturbations, obtained by linearising in $\varepsilon$, is $0.333$ compared with $0.157$ obtained above.

All the results reported here have been obtained by hand computation and it is unlikely that the problem will be studied any further by this method. Miss S. J. Roberts is at present coding the problem for an electronic computer; it is hoped that when this is done, more general field configurations and trial functions will be able to be considered. It is also hoped that further evidence will be obtained about the convergence of the variational procedure.

**Recent Developments in the Theory of the Stabilised Pinch**

Since the previous section was written it has been shown that the convergence problem mentioned in the last paragraph is a real one. Rosenbluth \textsuperscript{27} and Suydam \textsuperscript{28} have studied the mathematical character of the third Euler equation and have shown that, because of a singularity in this equation, the stabilised pinch is unstable against surface instabilities for any depth of penetration of the current. This singularity occurs at the radius in the plasma where the magnetic field helix coincides with the perturbation helix. It can be asked whether this singularity is physically real; in particular what happens to it when particle motions and dissipative processes are considered? A first attempt at a study of this problem has recently been made by Hubbard \textsuperscript{29} at Harwell.

Hubbard assumes that every particle is acted on not by a point magnetic field but by the field that has
been averaged over some averaging length ($\delta$) and he has replaced the singular term in the differential equation by its value averaged over this distance. He has solved the resulting differential equation on the Mercury computer, for many values of the averaging length and the depth of the current layer, and has found stability diagrams corresponding to Fig. 12a.

One such stability diagram is shown in Fig. 12b. In the limit of zero averaging length he has found results agreeing with those of Rosenbluth and Suydam. For finite values of the averaging length he has calculated the critical depth of field penetration at which stability ceases. He has also given arguments for choosing the averaging length and a discussion as to whether it is determined by the electrical conductivity, viscosity or particle gyration. He believes that the Larmor radius of the ions provides a suitable averaging length in many cases and his results sometimes show greater stability than the conventional stabilised pinch.

This work is at the moment based on physical intuition rather than on a rigorous mathematical foundation. He has merely modified the final second order differential equation and has supposed that marginal stability is still determined by the vanishing of the disturbance growth rate. In fact, if finite viscosity and electrical conductivity are included in the original equations, the order of the mathematical problem is considerably increased and complex growth rates occur.

**GENERAL STABLE DISCHARGE CONFIGURATIONS**

The conventional stabilised pinched discharge is not an obviously stable configuration; as has already been seen, very stringent conditions have to be satisfied by the parameters $b_1, b_2$ and $\Lambda$ for stability to be obtained. In the energy-principle formulation it is seen that the change in potential energy is positive in the vacuum and the centre of the plasma, and negative in the current layer and, for stability, an accurate balance has to be obtained. In this section the possible existence of obviously stable configurations is investigated. These are configurations in which the change in energy is positive in all parts of the discharge, for all perturbations, or even more restrictively ones in which all terms in the energy are positive everywhere. This problem has been studied at Harwell by Laing.\(^{30}\)

Suppose an ideally conducting plasma fills a tube of radius $\Lambda r_0$. If the plasma contains a magnetic field $(0, B_e, B_z)$, the change in its potential energy $W$ due to a perturbation $\xi e^{i(m\theta + k\zeta)}$, after minimisation with respect to $\xi_\theta$ and $\xi_\zeta$ is

$$\delta W = \int_0^{\Lambda r_0} r dr \left[ \left( \frac{mB_\theta}{r} + k B_z \right)^2 - 2B_\theta \left( B_\theta/r \right)' - 4B_\theta r^2 \xi'' \right],$$

where $\xi$ is the radial component of the displacement $\xi$ and differentiation is denoted by a prime. As the conducting fluid fills the entire tube, the radial perturbation $\xi$ must vanish at the radius $\Lambda r_0$. Using this condition, an alternative form can be obtained for $\delta W$ by performing an integration by parts on the term in $\xi''$ in (4.1).

Thus,

$$\delta W = \int_0^{\Lambda r_0} [A \xi'' + B \xi'''] dr,$$  \hspace{1cm} (4.2)

where

$$\begin{align*}
r A &= (mB_\theta + krB_z)^2 + (mB_\theta - krB_z)^2/(m^2 + k^2 r^2) \\
- 2B_\theta^2 - 2m^2 k^2 r^2 (B_\theta^2 + B_z^2)/(m^2 + k^2 r^2)^2 \\
- k^2 r^2 (B_\theta^2 + B_z^2)/(m^2 + k^2 r^2), \\
B &= r (mB_\theta + krB_z)/(m^2 + k^2 r^2).\end{align*}$$  \hspace{1cm} (4.3)

If the integrand in expression (4.1) is to be positive at all points in the discharge, the coefficient of $\xi''$ must be positive. Thus,

$$mB_\theta + krB_z > 2B_\theta (r B_\theta)'$$ \hspace{1cm} (4.4)

This condition has previously been obtained from the perturbed differential equations in Ref. 7. If Eq. (4.4) is to be satisfied for all perturbations m, k, the right-hand side of the equation must be negative or zero at all values of r. Thus,

$$I_z I_z' < 0,$$

where $I_z$ is the total axial current within radius r. This condition cannot be satisfied everywhere; in particular, at the discharge centre the current and its gradient must be in the same direction.

If the integrand in expression (4.2) is to be positive at all points in the discharge, $A$ must be greater than zero. Using the hydrostatic equation $\nabla p = j \times B / 4\pi$, this condition becomes

$$8\pi r p'd > a B_\theta^2 + 2b B_\theta B_z + c B_z^2,$$

where

$$a = m^2 - 1 + 1/(m^2 + X^2) + 2X^2/(m^2 + X^2)^2,$$

$$b = mX[-1 + 1/(m^2 + X^2)],$$

$$c = X^2[-1 - 1/(m^2 + X^2) + 2m^2/(m^2 + X^2)^2],$$

$$d = X^2/(m^2 + X^2),$$

$$X = kr.$$

(4.7)

It can be shown that the worst perturbation for the inequality (4.6) is that with $k = 0$ and $m = 1$. If the inequality is to be satisfied for this perturbation,

$$r p' > B_\theta^2 / 6\pi.$$

(4.8)

If the inequality (4.8) is satisfied at all values of the radius, the plasma cannot be contained by the magnetic field. Equation (4.8) has been obtained by minimizing the right-hand side of inequality (4.6) for all possible values of $B_\theta$, $B_z$.

There remains the possibility that the pressure can increase outwards near the centre of the discharge and that the current and its gradient are in opposite directions in the outer region. Thus $\delta W$ is divided up, so as to use the form (4.2) in the central region and the form (4.1) outside. There is also an additional term in $\delta W$, from the integration by parts on the interface $r = r_0$. Thus

$$\delta W = \int_{r_0}^{r_0} \left( A' \xi^2 + B' \xi^2 \right) dr$$

$$+ \int_{r_0}^{r_0} r dr \left( \left( \frac{mB_\theta}{r} + B_\theta \right)^2 - 2B_\theta (B_\theta / r)' \right.$$

$$- 4B_\theta \xi / r^2 + \left( \frac{mB_\theta}{r} - kB_z \right) \xi^2$$

$$\left. - \left( \frac{mB_\theta}{r} + kB_z \right) r \xi^2 \right) (m^2 + k^2r^2)^2$$

$$+ \left( \frac{(kB_z^2 - m^2B_\theta^2 \xi^2)}{m^2 + k^2r^2} \right)_{r = r_0},$$

(4.9)

It is now insisted that all parts of this expression are positive. If the surface term is to be positive for all m and k, $B_\theta$ must vanish on $r = r_0$; thus the total axial current inside $r = r_0$ vanishes. In addition, there can be no axial current outside $r = r_0$; otherwise the inequality (4.5) would once again fail. The final requirements are

$$r p' > B_\theta^2 / 6\pi,$$

(4.10)

If $B_\theta = 0$, $p' = -(B_\theta^2') / 8\pi$ for $r_0 < r < \Delta r_0$.

A completely stable configuration satisfying the conditions (4.10) has been found by Laing. The
values of magnetic fields and pressure for this configuration are shown in Fig. 13. It is not necessary for the axial magnetic field to drop to zero where the current has its maximum, as it does in this example; the example was obtained by assuming that $B_\phi$ has a simple analytical form. It is hoped that this type of configuration will be studied further and that neighbouring stable configurations can be found, which do not satisfy the very restrictive conditions of this section.

When the theoretical results of this section were obtained, it was not realised that Burkhardt and Lovberg\(^{31}\) had obtained a configuration similar in principle to the ones described above. In their case the reversed axial current arose unexpectedly and the discharge appeared to be stable.

**FURTHER PROBLEMS**

**Plasma Motions**

The only problem in which plasma motions have been considered in detail is that of the wriggling discharge. However, there is no reason to suppose that plasma motions are absent, even in an essentially stabilised discharge. Three such motions are: the initial contraction; motion of expansion or contraction during field diffusion; and steady oscillations, which may occur even for a stable discharge. The presence of these motions may lead to new types of instabilities. Many problems arising from the presence of equilibrium motions are essentially non-linear in the velocity; certainly, perturbations no longer grow or decay harmonically with time.

The oscillation problem can be studied in a linear way, as a first approximation. If it is imagined that a force is instantaneously applied to counteract the plasma acceleration, at the time in its motion when it is momentarily at rest, then effectively the problem is the same as that of a plasma supported against gravity by a magnetic field. A simple problem of this type has been studied by Kruskal and Schwarzschild.\(^5\) In their problem the plasma was very unstable, because there was no magnetic field in the plasma. In the presence of crossed magnetic fields, which are required to stabilise the initial configuration, many of the Rayleigh-Taylor types of instability are also removed.

The next approach to the oscillation problem considers the plasma to be oscillating in one mode and considers small time dependent perturbations on this. The study of the stability of such a system leads to differential equations of a generalised Hill's type. Some progress has been made on one simple problem along these lines. The problem of the stability of a plane plasma, which is executing rigid body oscillations between plane conducting walls, can be reduced, in a certain approximation, to the solution of two simultaneous Mathieu equations. This is still only a very poor approximation to the real problem, in which there may be interaction between very many modes on equal terms and the oscillation energy may be passing back and forth between them.

**Finite Dissipation**

It has been stated that, on qualitative grounds, it is plausible to treat the electrical conductivity as infinite in stability calculations. Similar arguments can be made in support of the neglect of viscosity and thermal conduction in some problems. However, even if numerically the importance of dissipative processes looks small, it must be realised that their introduction completely alters the mathematical character of the stability problem. One particular problem, in which viscosity may be important, has already been mentioned; in the energy principle method the lowest energy, for non-symmetrical perturbations, is obtained for a perturbation $\xi$ with an infinite velocity shear. As the energy can be made arbitrarily close to this minimum, with a perturbation with large but finite shear velocity, the result is probably virtually un-
altered for small enough viscosity; but this requires further investigation. For axisymmetric perturbations the trouble does not arise and, for sufficiently symmetrical systems and perturbations, a generalised form of energy principle still appears to be valid. A solution has been given to one problem involving viscosity in Ref. 6. For the earliest problem of the instability of the pinch effect studied by Kruskal and Schwarzschild\(^4\) there is an infinite instability growth rate at zero wavelength. The introduction of any viscosity, however small, reduces this growth rate to zero. This is illustrated in Fig. 14. An attempt is also being made to study the stability of a plasma, with finite electrical conductivity carrying a uniform axial current and a uniform axial magnetic field. With no axial field, the problem is simple; with an axial field, it is only easy to obtain asymptotic results and, at present, no general results on the effect of finite electrical conductivity have been deduced.

Two other factors are likely to become much more important to us in the future: the geometrical factor, which arises both through toroidal shape and through incomplete symmetry produced by gaps in conducting walls; and the problem of the range of validity of the hydromagnetic equations. It is true to say that the problem of gas discharge stability is still in its early stages and much remains to be done.

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