Simple Waves and Shock Waves in Magnetohydrodynamics

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As is well known, the equations of hydrodynamics are valid if the mean-free path \( l \) is small in comparison with the characteristic length \( L \) involved in the problem. For a plasma in a strong magnetic field, the hydrodynamic approximation can also be used even in those cases when this criterion is not fulfilled. This follows from the fact that in a strong magnetic field there is a second characteristic length \( R_L \), the Larmor radius. If this length is much less than the characteristic size of the system, \( R_L \gg L \), the equations of magnetohydrodynamics will be valid in the plane perpendicular to the magnetic field provided that the frequency of the field oscillation is much less than both the cyclotron frequency \( \omega_c = eH/Mc \) and the plasma frequency \( \omega_p = (4\pi n e^2 / M)^{1/2} \), which are characteristic frequencies of the plasma. The equations of magnetohydrodynamics can also be used to describe the motion along the magnetic field if the extent of the particle motion along the magnetic lines of force is finite. One should bear in mind that in this case, if collisions do not occur frequently, the pressure entering into the magnetohydrodynamic equations is anisotropic and depends not only upon the density but upon the magnetic field as well. Collisions, however, will tend to reduce the degree of the pressure anisotropy.

In deducing the magnetohydrodynamic equations from the equations of motion for electrons and ions, one usually supposes the validity of Ohm’s law and infinite conductivity. In reality, however, these assumptions are not required. It is possible to prove that in order to deduce the magnetohydrodynamic equations it is only necessary to suppose that the fields change slowly in space and time.

This paper gives some results of an investigation of non-linear plasma effects in the hydromagnetic approximation when the pressure is isotropic.

**SIMPLE MAGNETOHYDRODYNAMIC WAVES**

Generally speaking, non-linear equations do not admit solutions in which all the variables depend upon the quantity \( x - Vt \), because the wave form varies during its motion*; here, \( x \) is a space coordinate, \( t \) is the time and \( V \) is the constant phase velocity.

A natural generalization of this type of wave is the simple wave, introduced for the first time by Riemann, which has the property that all the hydrodynamic variables \( u_k \), \( k = 1, 2, \ldots, n \), are functions of one of them, for example \( u_1 \). The importance of simple waves is connected with the fact that in the absence of shock waves only simple waves can bound domains of constant flow. This statement, well known in the case \( n = 2 \), appears to be valid for any \( n \).

In this section we shall enumerate all the simple hydromagnetic waves and establish their connection with linearized plane waves.

We write the system of hydromagnetic equations schematically as

\[
\sum_{j=1}^{n} (X_{ij} \partial u_j / \partial x + T_{ij} \partial u_j / \partial t) = 0, \tag{1}
\]

where \( T_{ij} \) and \( X_{ij} \) are some functions of \( u_j \).

Let us consider first the linearized plane waves which correspond to small amplitude oscillations of the variables \( u_j \) about the constant value \( u_j^0 \). There are \( n \) types of such linear waves,

\[
\delta u_j = U_{j\mu} (u_1^0, u_2^0, \ldots, u_n^0) \exp (i(kx - \omega t)), \quad \mu = 1, 2, \ldots, n
\]

where \( U_{j\mu} \) is the wave amplitude, \( k = \omega / V_{\mu} \) is the phase velocity \( V \) is arbitrary, while the oscillation frequency is a complex function of the amplitude \( t \):

\[
\omega = \pi \omega_0 (1 - k^2)^{1/2} E - (1 - \omega_0^2) K
\]

where

\[
\omega_0^2 = 4 \pi n e^2 / m \]

\[
k^2 = \frac{1 - (1 - U_{\text{max}})^2}{1 + (1 - U_{\text{max}})^2} \]

\[
K = \int_0^1 \left( \frac{1 - x^2}{(1 - x^2)(1 - k x^2)} \right) dx
\]

\[
E = \int_0^1 \left( \frac{1}{1 + k x^2} \right)^{1/2} dx
\]

and \( U_{\text{max}} \) denotes the maximum value of \( v/c \), where \( v \) is the electron velocity. The quantity \( n_e \) is the electron density.

* Original language: Russian.

* Non-linear oscillations of a plasma at low pressures are an important example of such types of wave motion. In this case
wave vector, and \( V_\mu = V_\mu(u^\mu) \) is the phase velocity of the \( \mu \)th wave determined by the equation

\[
\det [X_\mu\nu(u^\rho) - V T_\mu\nu(u^\rho)] = 0. \tag{2a}
\]

To each linearized wave corresponds the simple wave determined by the equation

\[
\frac{du_1}{U_1} = \cdots = \frac{du_n}{U_n},
\]

\[
\hat{\partial}u_1/\hat{\partial}t + V_\mu(u_1, u_2, \ldots, u_n)(\hat{\partial}u_1/\hat{\partial}x) = 0. \tag{2b}
\]

The general solution of (2) is given by

\[
x - V_\mu(u_1, u_2, \ldots, u_n)t = f(u_1), \tag{3}
\]

where \( f(u_1) \) is an arbitrary function determined by the initial conditions. It follows that the displacement wave vector, and \( V_\mu \) is the phase velocity determined by the equation

\[
\frac{du_1}{U_1} = \cdots = \frac{du_n}{U_n},
\]

\[
\hat{\partial}u_1/\hat{\partial}t + V_\mu(u_1, u_2, \ldots, u_n)(\hat{\partial}u_1/\hat{\partial}x) = 0.
\]

The simple waves may now be enumerated:

1. Hydromagnetic waves:

\[
H_x^2 + H_z^2 = \text{constant}, \quad v_\rho = \pm V_y, \quad v_\varphi = \mp V_z,
\]

\[
q = \text{constant}, \quad s = \text{constant},
\]

\[
x - (v_x \pm V_x)t = f(H_z)
\]

where \( s \) is the entropy and \( \mathbf{V} = \mathbf{H}/(4\pi q) \) is the Alfvén velocity.

2. Magnetosonic waves:

\[
dv_2/dq = u/\sqrt{q},
\]

\[
dv_3/dq = -H_\varphi H_\rho/4\pi \sqrt{q}(u^2 - V_x^2),
\]

\[
dv_2/dq = -H_\varphi H_\rho/4\pi \sqrt{q}(u^2 - V_x^2),
\]

\[
dH_2/dq = u^2H_\rho/q(u^2 - V_x^2),
\]

\[
x - (v_2 \pm u)t = f(q),
\]

where \( u^2 = d(p + \frac{1}{2} \varphi V^2)/dq \frac{1}{2} \) is a root of the equation

\[
\frac{1}{2}v^4 + (V_y^2 + c_s^2)u^2 + c_s^2V_x^2 = 0,
\]

and \( c_s \) is the speed of sound propagation.

3. Entropy waves:

\[
x - v_2t = f(s),
\]

with \( \mathbf{v}, \mathbf{H}, q \) = constant.

An important property of simple magnetosonic waves is that their phase velocity depends upon quantities \( \mathbf{v}, \mathbf{H}, q \) which vary from point to point. Therefore these waves change their form during propagation. In this respect they are different from simple hydromagnetic waves which have a constant phase velocity and therefore propagate without changing their form.

Equation (5) shows that the phase velocity of a simple magnetosonic wave increases with increasing density if

\[
(\partial^2 /\partial t^2) > 0.
\]

This condition is valid for most substances.

Since similarity waves are a particular case of simple magnetosonic waves they are always rarefaction waves. Another important consequence of Eq. (7) is that in the compression region the density gradient increases and this leads to the formation of shock waves.

**POSSIBLE TYPES OF SHOCK WAVES**

We now employ the general equations (1) to investigate the stability of plane hydromagnetic shock waves subjected to small disturbances which depend only upon the distance from the surface and the time.

Shock waves will be instable if the total number of hydromagnetic, magnetosonic and entropy waves propagating from the discontinuous front is not equal to six.

To prove this statement, let us denote by \( u^{1,2}_{\mu} \), the magnitudes of the variable \( u_\mu \) on both sides of the surface of discontinuity. The index 1 corresponds to the region ahead of the shock front, \( x < 0 \), and the index 2 corresponds to the region behind the front, \( x > 0 \). Let us also describe the perturbation by \( \delta u^{1,2}_{\mu}(x, t) = u^{1,2}_{\mu}(x, t) - u^{1,2}_{\mu} \). If the discontinuous solution \( u^{1,2}_{\mu} \) is stable, these variables will stay small if they are small initially. In this case the linearized equations

\[
\sum_{j=1}^{n} \{X_{ij}^{1,2}\delta u_{ij}(x, t)/\partial x + T_{ij}^{1,2}\delta u_{ij}(x, t)/\partial t\} = 0 \tag{8}
\]

are valid where

\[
X_{ij}^{1,2} = X_{ij}^{1,2}(u^{1,2}), \quad T_{ij}^{1,2} = T_{ij}(u^{1,2}).
\]

It may happen that this system of equations does not have a unique solution. On the other hand, Cauchy’s problem in hydrodynamics must always have a unique solution. Therefore, the absence of a unique solution to the system (8) indicates that it is impossible to replace the exact equations with linearized equations, and this in turn implies that perturbations which are small initially will not stay small at later times.

A discontinuous change in \( \delta u^{1,2}_{\mu} \) in time occurs only if the initial shock wave splits into several waves in the regions enclosed between the newly formed surfaces of discontinuity. Therefore, the criterion for the stability of a shock wave is the existence of a unique solution for the linearized Cauchy problem (8) corresponding to the system of Eq. (1).

The well-known general solution of Eq. (8) which satisfies the initial conditions is

\[
\delta u^{1,2}_{\mu}(x, t) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} C_{n}^{\mu,2}(\omega, x)e^{i(\omega_{\mu}-\omega)t} d\omega, \tag{9}
\]
where \( k_1^{1.2} = \omega/V_1^{1.2} \) and \( \mu \) enumerates the various plane waves. The integration is carried out in the upper half-plane along the straight line parallel to the real \( \omega \) axis, \( \text{Im} \omega > 0 \). The amplitudes \( C^{\mu, 1.2}_j(\omega, x) \) and \( C^{\mu, 2.1}_j(\omega, x) \) are obtained by the method of variation of constants; they satisfy a system of \( n \) inhomogeneous differential equations. Therefore there are \( n \) arbitrary constants in the expression (9) for \( \delta u_1 \) and \( \delta u_2 \). These constants must be selected so that the boundary conditions at the discontinuity surface are satisfied and also so that \( \delta u_1^2(-\infty, t) \) and \( \delta u_2^2(\infty, t) \) are finite.

Consider now the latter condition in detail. If the linearized wave with index \( \mu \) moves to the right, \( (V_\mu > 0) \), then \( \text{Im} k_\mu \) must be positive so that the perturbations approach zero as \( x \rightarrow -\infty \). If the wave propagates to the left \( (V_\mu < 0) \) then the imaginary part of \( k_\mu \) will be negative and in order for \( \delta u_1^2(x, t) \) to be finite as \( x \rightarrow +\infty \), it is necessary to set equal to zero the amplitudes \( C^{\mu, 1.2}_j(\omega, x) \) that correspond to waves converging toward the discontinuity front. These conditions must also be applied on the opposite side of the discontinuity surface. If the total number of waves dispersing in both directions is equal to \( m \), the number of convergent waves will be \( (2n - m) \). Adding the boundary conditions for the discontinuity surface to \( (2n - m) \) conditions which must be placed on the \( 2n \) arbitrary constants contained in \( C^{\mu, 1.2}_j(\omega, x) \). In order to obtain a unique solution to the Cauchy problem, it is necessary to have the number of conditions, \( 3n - m \), equal to the number of unknown quantities, \( 2n + 1 \). The unknown quantities consist of \( 2n \) integration constants and the perturbed velocity of the discontinuity front. The necessary condition for stability is therefore

\[
m = n - 1;
\]

\[(10)\]
i.e., the number of divergent waves must be one less than the number of boundary conditions at the discontinuity surface. If this condition is met, one must still determine whether it is possible to satisfy the boundary conditions with the available arbitrary constants.

In magnetohydrodynamics there are seven types of plane waves, which can be classified according to their phase velocities \( v_x \pm V_x \) (hydromagnetic waves), \( u_x \pm u_\perp \), where \( u_\perp \) are positive roots of Eq. (6) (magnetosonic waves), and finally \( v_y \) (entropy waves).

A necessary condition for the stability of a shock wave, according to Eq. (10), is that the number of divergent waves be equal to six. Taking into account the inequality \( u_x^{1.2} < V_x^{1.2} < u_x^{1.2} \), we find that a shock wave can be stable only under the following three conditions:

A. \( u_x^{1.2} < v_x < u_x^{1.2} \);

B. \( V_x^{1.2} < v_x < u_x^{1.2} \);

C. \( u_x^{1.2} < v_x < V_x^{1.2} \).

These conditions are displayed in the stability diagram of Fig. 1.

If the magnetic field is parallel to the discontinuity front these conditions are reduced to the inequalities

\[
(v_x^{1.2})^2 > c_1^2 + (V_x^{1.2})^2, \quad (v_x^{1.2})^2 < c_2^2 + (V_x^{1.2})^2
\]

(see Fig. 2).

When the magnetic field is perpendicular to the front there are additional considerations since then the boundary conditions split into three isolated groups. The stability regions for this case are shaded in Figs. 3a, 3b and 3c.

The necessary conditions for stability which have been obtained make it possible to carry out the classification of shock waves. In the general case, when the magnetic field is inclined relative to the plane of discontinuity, there are only three types of stable waves which correspond to the three regions A, B, and C in Fig. 1. If the jumps in all the variables at the discontinuity approach zero, the shock velocities approach the phase velocities of the linearized waves. Furthermore, in this limit, the velocity of a type A wave approaches the phase velocity \( v_x \) of the magnetosonic wave and the velocities of types B and C waves approach the velocities \( v_x \pm V_x \) and \( v_x \pm u_\perp \) of the hydromagnetic and magnetosonic waves, respectively. In this connection, it is expedient to subdivide the shock waves into slow magnetosonic, hydromagnetic, and fast magnetosonic waves.

If the magnetic field is parallel to the plane of discontinuity, then only fast magnetosonic shock waves can be stable. If the magnetic field is perpendicular
Figure 3 a, b, c. Stability diagrams when the magnetic field is perpendicular to the discontinuous front

to the discontinuity, only magnetosonic discontinuities can be stable (see Figs. 3a, 3b, 3c). Let us now note some consequences of the stability conditions for shock waves.

1. If two shock waves of the same type follow each other, the last wave will always catch up to the first wave.

2. If two waves of different types follow each other, a hydromagnetic shock wave will catch up to a slow magnetosonic wave while a fast magnetosonic wave will catch up to any shock wave.

3. A shock wave will catch up to a weak discontinuity of the same type as the shock wave or of a slower type.

4. A weak discontinuity will catch up to a shock wave of the same type or of slower types.

IMPOSSIBILITY OF RAREFACTION SHOCK WAVES IN MAGNETOHYDRODYNAMICS

In ordinary hydrodynamics, in conformity with Cempen’s theorem, rarefaction shock waves are impossible if

$$\frac{\partial^2 (\rho^{-1})}{\partial \beta^2} > 0.$$  

In magnetohydrodynamics, shock waves are also compression waves if the condition

$$\frac{\partial \rho}{\partial T} > 0$$

is satisfied together with condition (11). Cempen’s theorem in magnetohydrodynamics follows from the fact that the shock adiabat in the \((\rho, q)\) plane in the presence of a magnetic field is located below the shock adiabat in the absence of a magnetic field, for \(q_2 < q_1\). This section of the adiabat \(q_2 < q_1\) cannot exist since the entropy must decrease, \(s_2 < s_1\) (see Fig. 4).

Figure 4. Shock adiabat in the \((P, Q)\) plane

From the boundary conditions at the discontinuity it follows that

$$H_+^2 = H_-^1 = q_2 \frac{(v_2)^2 - (v_1)^2}{q_1 (v_2)^2 - q_2 (v_1)^2},$$

where \(H_+\) is the projection of the magnetic field on the discontinuous surface. This equation leads us to the following conclusions:

1. If \((v_2)^2 < (V_2)^2\), the tangential component of the magnetic field does not change its direction and \(|H_+^2| < |H_-^1|\).

2. If \((V_2)^2 < (v_2)^2 < 2q_2(V_2)^2/(q_1 + q_2)\), the tangential component of the magnetic field changes its direction and \(|H_+^2| < |H_-^1|\).

3. If \(2q_2(V_2)^2/(q_1 + q_2) < (v_2)^2 < q_2(V_1)^2/q_2\), the tangential component of magnetic field changes its direction and \(|H_+^2| > |H_-^1|\).

* De Hoffman and Teller have shown that a compressional shock wave in a perfect gas is thermodynamic stable if the magnetic field is parallel to the discontinuity surface. The question of the thermodynamic instability of a rarefaction shock wave was not decided.
Finally, if \( q_3(V_x^2)^3/q_1 < (v_x^1)^2 \), the tangential component of magnetic field does not change its direction and \( |H_{||}| > |H_{\perp}| \).

These inequalities demonstrate that a shock wave passing through a weak field \( H_x < [2\pi (q_1 + q_3)/q_2]^1/2v_x^1 \) intensifies the magnetic field, while a shock wave passing through a strong magnetic field for which this inequality is reversed, reduces the strength of the magnetic field. This demonstrates that shock waves have some smoothing action when passing through a magnetic field.

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