Mechanization of Non-linear Calculations in Fusion Reactor Theory

By P. Braffort and M. Chaigne *

From Kurchatov's report and recent British publications, the physical foundations of fusion reactor operation have gradually been emerging. Before attacking the technological problems presenting themselves to the engineer, it is necessary to specify various quantities roughly, and to indulge in some (inevitably) very approximate calculations for that purpose. The object of our paper is to emphasize the role of automatic computing, in particular analogue computing, in this field. Thus the study of plasmas and the phenomena they involve calls for solution of a large number of differential, integro-differential and partial differential equations which are all but impossible to attack manually. The analogue computation that has afforded a quick and accurate survey of the kinetics and dynamics of a wide variety of fission reactors lends itself also to the solution of plasma equations, despite their essentially non-linear character.

For example, the purely dynamic problem of the evolution of a high current discharge in a plasma involves a "resistivity" that depends on the temperature raised to the $3/2$ power. It is therefore necessary to use a generator of functions of the type $y = x^n$. Upon entering the space-time field, the conventional use of non-linear elements in analogue computation no longer suffices. But the underlying principle of the analogue method is still applicable (that of generalized negative feedback). At the cost of using certain approximations and complicating the automatic computation somewhat in structure, it is possible in this way to attack the space-time field of partial differential equations or integral equations. We shall offer some suggestions for the application of these methods to problems concerning the physics of fusion reactors.

MATHEMATICAL ASPECTS OF THERMONUCLEAR PROBLEMS

Analysis of thermonuclear phenomena takes place simultaneously on three levels, a fact which we shall have occasion to use: the molecular and atomic (ionic) level, the macroscopic level, and the nuclear level at which the ultimate energy-producing phenomenon occurs. The various interactions can be described as in Fig. 1.

In this section, we shall exhibit some equations or systems of equations characteristic of various phenomena encountered at all three of these levels, of course without any claim to completeness, and we shall try to bring out certain features they have in common from the point of view of mathematical analysis. In later sections, some of these problems will be taken up and methods of analogue solution suggested.

Nuclear Level

The effect of temperature on the movement of ions appears at the nuclear level and, in turn, affects the probabilities of internuclear reactions, which is to say the effective reaction cross sections and power densities. For DD reactions, the reaction rate $R$ is:

$$R = \frac{1}{4} n^2 \langle \sigma v \rangle_{av} \text{ reactions per cm}^3 \text{ per sec}$$

where $n$ is the density; $\sigma$, the effective reaction cross section; and $v$, the ions' velocity.

With a Maxwellian velocity distribution at temperature $T$ (kev), we have

$$\langle \sigma v \rangle_{av} = 260 \times 10^{-16} T^{-1} \exp (-18.76 T^{-1}); \quad (T < 50 \text{ kev}).$$

This relation, we need hardly mention, is typically non-linear.

Atomic Level

On the atomic level, the movement of ions in the plasma is determined by Lorentz forces of the type

$$\mathbf{y} = \sigma[\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B}]$$

to which correction terms must be added, in particular a term in $\mathbf{y} \times \mathbf{B}$ due to the Hall effect, whence

$$\mathbf{y} = \sigma \left[ \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{eNc} \mathbf{y} \times \mathbf{B} + \frac{1}{eN} \nabla \phi \right]$$

where $\sigma = \text{conductivity}$, $\mathbf{E} = \text{electric field}$, $\mathbf{B} = \text{mag-
netic field, \( \mathbf{v} = \) displacement velocity and \( c = \) velocity of light.

We now have a classical dynamics problem, provided, of course, we know \( \mathbf{E} \) and \( \mathbf{B} \).

**Solution of Field Equations**

To find the field \( (\mathbf{E}, \mathbf{B}) \) one must solve partial differential equations of the Maxwell type:

\[
\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E}
\]

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{H}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{J} \times \mathbf{H}_0
\]

\[
\frac{4\pi}{\omega_0^3} \left( \frac{\partial J}{\partial t} + \gamma J \right) = E + \frac{1}{c} \mathbf{v} \cdot \mathbf{H}_0 - \frac{m_i m_o}{e} \frac{\partial \mathbf{v}}{\partial t}
\]

where \( \mathbf{H} = \) magnetic field, \( \mathbf{E} = \) electric field, \( \mathbf{J} = \) current density, \( \omega_0 = \) natural frequency of plasma, \( \gamma = \) frequency of interaction of particles, \( m_i = \) velocity of ions, \( m_o = \) velocity of electrons, and \( p = \) pressure. These are equations to which the analogue method can be adapted only with the aid of certain approximations, but such approximations are now entirely available.

**Macroscopic Level**

The electromagnetic field governing the movement of ions is not generated solely by the external circuits acting within cavities of definite shape, defining well-known boundary conditions. It also results from the movement of the plasma ions themselves so that special phenomena such as the pinch effect arise. Here we have the transition from the atomic to the macroscopic level. This transition can be taken into account by considering the statistical motion of individual ions. Mathematically, we obtain equations of the Boltzmann type. In the case of a Lorentzian plasma subjected to a constant magnetic field \( H \) and an alternating electric field \( E \cos \omega t \), the transport equation is:

\[
Df(x,v,t)/Dt = J(f),
\]

where the operator \( D/Dt \) is given by:

\[
D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla_x + (\mathbf{\omega} \times \mathbf{v}) \cdot \nabla_y
\]

in which \( \mathbf{\Gamma} = (e/m)\mathbf{E} \) and \( \omega_0 \mathbf{H} = (e/m)\mathbf{H} \). The function \( f(v,t) \) is the distribution function for electrons with velocity \( v \), and \( J \) is the Boltzmann collision operator applied to a Lorentzian gas; it takes into account only elastic electron-molecule impacts. There is a classical method of arithmetical calculation for the solution of such problems, the Monte Carlo method. We shall see how a procedure derived from that method can be used in the field of analogue computing.

The nuclear reaction rate is determined by the temperature of the plasma and by the particle density. The temperature corresponds to a mean kinetic energy and can therefore be obtained by combining the calculations for the atomic and macroscopic levels. A connection with the nuclear level is made by taking account of the control of the reactor. This will be done in a later section of the paper.

**EXAMPLES OF SOLUTIONS**

**A Purely Dynamic Problem**

We have observed that the motion of an ion in the electromagnetic field is an example of a dynamic problem. Ultimately, the equations are of the form \( \mathbf{F} = ma \), whence:

\[
\frac{d^2x}{dt^2} = F_x/m, \quad \frac{d^2y}{dt^2} = F_y/m, \quad \frac{d^2z}{dt^2} = F_z/m
\]
This is a system of three second-order differential equations, for which an analogue can easily be given (Fig. 2). The symbols in the analogue diagrams of this paper are given in Fig. 3.

Since the field, in this instance, is not given in advance and is not independent of time, it is necessary to solve other space-time problems before getting to this stage; we shall therefore defer discussion to a later section. Instead, as an example of a purely dynamic problem, we shall study the evolution of discharge current as a function of time in a fully ionized gas enclosed in a toroidal chamber. The problem reduces to the coupling of two electric circuits, one of which is the gas in the toroidal chamber. The system of equations to be solved is therefore:

\[ v_1 = R_1 i_1 + L_1 di_1/dt + M d^2 i_1/dt^2, \]
\[ v_2 = R_2 i_2 + L_2 di_2/dt + M d^2 i_2/dt^2, \]
\[ di_1/dt = -i_1/C_1 \quad \text{and} \quad di_2/dt = -i_2/C_2 \]
where the circuit diagram is shown in Fig. 4. From this, by introducing the damping coefficients and the characteristic frequencies of these circuits,

\[ a_1 = R_1/2L_1 \quad \text{and} \quad a_2 = R_2/2L_2, \]
\[ \omega_1 = 1/(L_1 C_1)^{1/2} \quad \text{and} \quad \omega_2 = 1/(L_2 C_2)^{1/2}, \]

we obtain the following system of equations:

\[ d^2 i_1/dt^2 + 2a_1 di_1/dt + \omega_1^2 i_1 = -(M/L_1) d^2 i_1/dt^2, \]
\[ d^2 i_2/dt^2 + 2a_2 di_2/dt + \omega_2^2 i_2 = -(M/L_2) d^2 i_2/dt^2. \]

The difficulty of such an investigation lies in the fact that the resistance of the gas is a function of the temperature and varies with time.

Supposing that the inductance \( L_2 \) is constant, using the formula given by Spitzer to compute \( R_2 \), and neglecting radiation losses, the term \( a_2 \) is finally given by

\[ a_2 = 2.3 \times 10^4 \left( \int i_2^2 dt \right)^{-1/2}. \]

Hence we have to solve the integro-differential system

\[ d^2i_1/dt^2 = -2a_1 di_1/dt - k (L_1^2/L_1) d^2 i_2/dt^2, \]
\[ d^2i_2/dt^2 = -4.6 \times 10^4 di_2/dt - k (L_1^2/L_2) d^2 i_1/dt^2. \]

The coupling coefficient \( k \) is defined by \( k^2 = M^2/L_1 L_2 \).

We immediately see the non-linear character of such a system. Also, the duration of the physical phenomenon is very short, the investigation extending over only 100 to 200 \( \mu \)sec. On the other hand, the intensity of the secondary current is very high and may reach a maximum of \( 10^6 \) amp. For analogue solution of this system, we shall therefore expand the time scale so as to study the phenomenon in about 2 minutes.

The \( \int i_2^2 dt \) term will be given by a servo-mechanism function generator in which an \( x^2 \) curve is introduced and which has a rotational speed governed by time.

We suggest two analogues (Figs. 5a and 5b), the choice depending on technical considerations (whether a divider is available or not).

This method of calculation will quite readily yield by direct recording the movement of discharge current for very different values of the coefficients, these being allowed to vary by a factor of 10. As to the accuracy of the results obtained, this depends on the precision of the recorder used.

**A Purely Spatial Problem**

In this section, we shall investigate the ionic density distribution \( n = f(r) \) in a fully ionized cylindrical plasma in an axial electric field \( E_z \) and an axial magnetic field \( B_z \), as a function of the distance \( r \) from the axis of the cylinder, as shown in Fig. 6.

In practice, the study of such a distribution requires knowledge of the behavior of the longitudinal magnetic field.
field \( B_z \) and the circular magnetic field \( B_\theta \) as a function of the distance \( r \) from the axis of the cylinder. These two fields are determined by the following two differential equations:

\[
\frac{dB_z}{dr} = 4\pi \frac{B_\theta B_{2z}}{B_{2z}^2 + B_{2z}^2},
\]

\[
\frac{dB_\theta}{rs} = 4\pi \frac{2 B_{2z}^2 + B_{2z}^2}{B_{2z}^2 + B_{2z}^2}.
\]

In these two equations, the unknowns are the normalized dimensionless variables

\[ B_{2z} = B_\theta/B_{20}, \quad B_{2z} = B_z/B_{20}, \quad \text{and} \quad r_s = r E_2 \sigma_1/B_{20} \]

where \( B_{20} \) is the initial value of the longitudinal magnetic field \( B_z \), or in other words, the value of \( B_z \) for \( r = 0 \); \( B_\theta \) and \( B_z \) are functions of \( r \); and \( \sigma_1 \) is the perpendicular conductivity of the plasma. Computation with these dimensionless variables will yield results that can be used regardless of the initial conditions.

To this system we may add the equation

\[
\frac{dn_s}{dr_s} = -B_{2s} dr_s
\]

giving the ionic density distribution in terms of the dimensionless variable

\[ n_s = 2nkT/B_{20}^2, \]

in which \( n = \) ionic density, \( k = \) Boltzmann’s constant, and \( T = \) absolute temperature of plasma.

The problem so stated is to be solved in a stationary case. Note that, in this instance, the plasma is symmetrical with respect to two perpendicular planes passing through the axis of the cylinder. We can therefore convert the spatial problem to a plane problem and confine ourselves to calculation along a radius.

To handle this problem simply on our machines, we shall take time as the analogue of distance from the axis of the cylinder. Thus all quantities that are functions of distance from the axis will become functions of time. In the present case, the movement of these various quantities over unit “normalized distance” will correspond to the behavior of the corresponding quantities during 10 seconds. Fair precision is thus obtainable by this method.

For this problem we also suggest an analogue diagram (Fig. 7), according as dividers are or are not available. Note, in passing, the simplification of this calculation by use of an “electronic operator”, serving to perform an operation of the form \( AB/C \). Elimination of a considerable number of amplifiers (several with high gain) and avoidance of division by means of a reciprocal operator substantially improve the accuracy of the calculation, which is limited by the precision of the recorder.

Analogue computation will also permit the handling of the inverse problem; that is, to find the value of a parameter (\( E_z \) for example) required for a given value of the ionic density \( n \) at a distance \( r \) from the axis. To solve this problem, we can proceed by successive approximation, systematically manipulating all the potentiometers in which the parameter in question is concerned. The procedure will then be to perform
MECHANIZATION OF FUSION CALCULATIONS

Figure 6. Analogue diagram for the determination of the ionic density distribution in an axial electric and magnetic field vs. the distance from the axis.

Figure 7. Analogue diagram for investigation of ionic density distribution.
the computation, in each case, for a time \( t \) corresponding to the distance \( r_s \), so as to obtain a smaller and smaller error relative to the fixed value of \( n \).

A Simple Space-time Problem

We here attack the problem of motion of a particle in an electromagnetic field. Let us take the case of a cylindrical geometry. Using the classical notation, the equations giving us the position \( z \) of the particle along the axis of the cylinder, the deflection \( r \) from that axis and the velocity \( dz/dt \) (and hence the energy) are

\[
m \frac{dz}{dt} = e E_z(z) \cos (\omega t + \phi)
\]

\[
E_r(z) = - \frac{r \partial E_z(z)}{2 \partial z}
\]

\[
m \frac{d^2 r}{dt^2} = e E_r(z) \cos (\omega t + \phi).
\]

As we said before, in such a problem the difficult part is to provide a function generator that will obtain the values of \( E_z \) and \( \partial E_z/\partial z \) at any moment. Since the analytical expressions do not lend themselves readily to computation, we have adopted approximate representations for \( E_z(z) \) and \( \partial E_z/\partial z \). To generate these functions, an apparatus has been devised, as shown diagrammatically and pictorially in Figs. 8 and 9. The two “Plexiglas” discs measure about a meter in circumference.

Each zone is divided into several sectors with conductive coating, marked with the potentials supplied by a network of resistors. The over-all analogue diagram is shown in Fig. 10.

Obviously such a device does not permit very great precision and, moreover, applies only to discontinuous geometries. Accordingly, a field simulator of more general usefulness has been devised. This field simulator automatically provides the solutions
MECHANIZATION OF FUSION CALCULATIONS

253

Figure 9. Pictorial sketch of disc function generator (servomotor not shown). 1 = Pick-up contacts; 2 = Silvered sectors with 1 mm separation; 3 = " Plexiglas"; 4 = Dull finish " Plexiglas"; 5 = 1% resistors; 6 = + 80 volts; 7 = — 80 volts; 8 = Shaft connected to motor; 9 = Metal disc attaching wheel to shaft; 10 = Second disc with 36 sectors; 11 = First disc with 28 sectors; 12 = Ground

Figure 10. Analogue diagram for an investigation of the motion of an ion in an electromagnetic field

of the potential equation in a two-dimensional space. It can also give the solutions of partial differential equations of various kinds and do this for any boundary conditions, even those dependent on time.

The principle of this field simulator is the following: Let \( (x, y) \) be a plane space within which arbitrary areas \( S_1, S_2, \ldots, S_n \) are subjected to potentials \( V_1, V_2, \ldots, V_n \). These potentials may vary with time. For any moving point \( M \) with coordinates \( x(t) \) and \( y(t) \), the values \( \partial V/\partial x \) and \( \partial V/\partial y \) are required.

For this purpose, let us make a small-scale model of the space in question (see Fig. 11).

We draw the outlines of the areas \( S_1, S_2, \ldots, S_n \), cover them with conductive coating and bring them to potentials \( k_1V_1, k_2V_2, \ldots, k_nV_n \) on suitable intersections, thus setting up boundary conditions.

Now the potential difference between two successive intersections in the \( x \) direction is proportional to \( \partial V/\partial x \) and the potential difference between two successive intersections in the \( y \) direction is proportional to \( \partial V/\partial y \).

Hence, if we require a potential reader to follow the \( x(t) \) and \( y(t) \) law, it will at any time deliver two signals, one proportional to \( \partial V/\partial x \) and the other to \( \partial V/\partial y \), where \( V = V(x(t), y(t)) \).

SOLUTIONS OF SPACE-TIME PROBLEMS IN THE GENERAL CASE

Though the investigation of the individual movements of charged particles in the plasma is very interesting, particularly for the study of various limiting cases, it will not yield an over-all conception of the phenomenon because the logical " loop " relating the phenomena diagrammed in Fig. 1 is not closed. Hence it is necessary to attack the macroscopic problem of the plasma.

The underlying phenomena, as we have seen, are the following: individual movements of ions in the field between two impacts; impacts which cause ion diffusion; and generation of fields by the motion of the ions. Expressed in mathematical form, this leads to two systems of equations: (a) diffusion equations, derived from an equation of the Boltzmann type, and (b) field equations derived from Maxwell's equations.

We are therefore confronted with a system of partial differential equations so complex that the method described above is not directly applicable. Numerical calculation could of course be used, but here again the complexity of the system to be solved would render the programming very extensive and perhaps beyond the memory capacity of ordinary machines.

Now there is an approximate method of calculation currently used in numerical computation to avoid excessive memory requirements—the Monte Carlo method. The principle is familiar; to compute an integral of the form \( \int f(x)g(x)dx \), we regard \( f(x) \) as a probability density, find a random variable \( x \) having...
Figure 11. Scale model as discussed in text

a density of this type, and take means over randomly chosen populations of values of $x$, or rather $g(x)$.

This method is especially convenient for solving diffusion equations. An interesting fact may here be noted: the transition from an equation of the type

$$\frac{\partial P}{\partial t} = K \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right)$$

to an equation of the type

$$P(m, n, s + 1) - P(m, n, s) = \frac{1}{2} \left[ P(m + 1, n, s) - 2P(m, n, s) + P(m - 1, n, s) \right] + [P(m, n + 1, s) - 2P(m, n, s) + P(m, n - 1, s)],$$

defined over a two-dimensional grid, is equivalent to the transition from the macroscopic transport equation to the microscopic equation describing the "random walk" of individual molecules. (This is only natural; for the inverse transition was precisely that used by Chapman-Kolmogoroff in setting up their diffusion equations.)

This tells us what route to follow, a route which we shall here describe only in very broad outline. Instead of taking as random phenomenon a discontinuous Brownian movement with transition probabilities given by a table of random numbers, we shall make direct use of simulation of the motion and collision of individual particles, of which we shall take the means. Each individual calculation will be performed as follows. The motion of an ion will be integrated in the manner outlined in the previous section, but:

(a) the initial values for position and velocity will be given by a noise generator; and

(b) the duration of the computation will likewise be determined by chance with the aid of a noise generator (with mean value corresponding to the mean-free path).

The results of the several calculations will be placed in a memory circuit and averaged periodically; they will be used to compute the macroscopic velocity and current according to the formulas:

$$\mathbf{v} = (n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e)/Q$$

$$\mathbf{j} = (n_i Z \mathbf{v}_i - n_e \mathbf{v}_e) e/c.$$

The current $\mathbf{j}$ will, in turn, be put into the device in order to solve Maxwell's equations.

Certain technical problems arise at this point. Provision must be made for memory storage of certain results and for their delivery at definite times. We have constructed an apparatus, meeting these requirements, which will be described elsewhere. Also, it will be necessary to make individual calculations on a time scale differing from the scale used for the simulator taken as a whole. We have replaced the space averages by time averages, in effect, by making use of the ergodicity of the phenomena considered, and these averages taken successively must not too greatly affect the behavior of the whole. Here again, multiple-time simulation techniques have been developed.

Finally, the phenomena connected with the electromagnetic field take place in a three-dimensional geometry. This leads to the use of not one but two field simulators of the type previously described, suitably coupled together. The general diagram of such a simulator is given in Fig. 13; a more detailed description will be published elsewhere.

Our proposed device has the advantage of being based on simulation of the elementary physical phenomena themselves and thereby avoids simplifications whose macroscopic effect is difficult to ascertain in such a highly non-linear field. Its disadvantage is that it cannot function in real time and hence does not correspond to real thermonuclear engines. It appears to us that this disadvantage should not prevent our machine from finding a wide field of application, even outside the domain of plasma physics.

ROLE OF NON-LINEARITIES IN THE OPERATION OF A THERMONUCLEAR REACTOR

Though the solution of engineering problems involved in the harnessing of thermonuclear energy is far from complete, it is not impossible to investigate immediately some of the dynamic problems presenting themselves in this field.

Schultz, taking his departure especially from Post's work, has proposed a system of dynamic equations expressing the balance of energy and the numerical balance of particles within a reactor, respectively. These equations are as follows:
MECHANIZATION OF FUSION CALCULATIONS 255

The source $S$ here represents the supply of fuel to the reactor. It is then possible to compute the Laplace transforms of this linear system and all the transformation factors, $S \to T$ and $S \to n$, and then find the conditions of stability by using the familiar criteria of servo theory. As these equations are highly non-linear, it may be wondered whether Schultz’s linearization may not pass over some regions of instability. We have therefore undertaken some attempts to simulate the non-linear equations. Construction of the analogue circuit presents some difficult problems when it is desired to minimize the number of amplifiers and multipliers. Figures 14a and 14b show two equally possible set-ups, to be adopted according to technical considerations (whether servomechanism multipliers or electronic multipliers are available).

For some values of the coefficients, instabilities are actually observed.

Of course, the formulation proposed by Schultz involves drastic simplifications whose implications are difficult to assess. But, in any case, the foregoing inquiry has the merit of emphasizing the danger of linearization in this field. It may be inferred that simulation is here absolutely indispensable.

**CONCLUSION**

We have presented a group of methods affording a considerable extension of the field of application of analogue computing to theoretical studies concerning fusion reactors. In all the cases considered, of course, methods of numerical computation are possible, but seem to be avoidable because:

1. The precision required is not high, considering that the physical quantities involved in the equations are not accurately known; and
2. at the present stage of inquiry, it is essential to vary all parameters within very wide limits, which is especially convenient by the analogue method.

At any event, the methods described permit one to avoid linear approximations, thereby pointing directly to certain points of difficulty in the systems studied and preparing the way for more thorough investigation in the future.
REFERENCES

1. I. V. Kurchatov, Bulletin d'informations scientifiques et techniques, 4 (1956) (French translation)
13. Charron, private communication
14. L. Spitzer, Physics of Fully Ionized Gases (1956)
15. Charron, unpublished report
17. J. Gamp and M. Deat, CALA Report, C.E.A., France
20. G. W. Brown, Monte Carlo Methods, Modern Mathematics for the Engineer, 279 (1956)
22. J. Gamp, ACLA Report 23 (forthcoming)
24. Schulz, Nucleonics, 15, 6, 76 9 (1957)