Thermonuclear Reaction Rate in a Discharge

By H. Grad

The electrical conductivity of a plasma is found by computing the deviation from Maxwellian of the electron distribution produced by an externally applied electric field. This problem has been solved for the Boltzmann equation using the Chapman-Enskog theory by Landshoff. In principle, the same methods could be used to compute the ion distribution and from it the thermonuclear output. However, for this application, the high-velocity tail of the ion distribution is of crucial importance and this cannot be evaluated by the conventional methods. The reason lies in the method of linearization employed. The electric field \( \mathbf{E} \) is assumed to be small, and the resulting deviation of the distribution \( f \) from the Maxwellian \( f^0 \) is also small. The "forcing" term \( \mathbf{E} \cdot \partial f/\partial \varepsilon \) is replaced by

\[
\mathbf{E} \cdot \partial f^0 / \partial \varepsilon = \mathbf{E} \cdot \xi f^0 / RT
\]

(see Eq. (12)). This is a small perturbation of a Maxwellian \( f^0 \) only if \( \mathbf{E} \cdot \xi \) is small, not when \( \mathbf{E} \) alone is small. Furthermore, the collision term, which is a quadratic functional of \( f \) in either the Boltzmann or the Fokker-Planck equation, is usually linearized according to the scheme \( f = f^0 + f^1 \), so that,

\[
Q(f, f) \rightarrow Q(f^0, f^0) + Q(f^0, f^1) + Q(f^1, f^0),
\]

assuming that \( f^1 \ll f^0 \). We shall find the perturbation \( f^1 \) to be much larger than \( f^0 \) in the tail of the distribution even though \( f^1 \) is always small in an absolute sense.

We shall be able to avoid these difficulties by making use of certain special features of the Fokker-Planck equation and by introducing an asymptotic expansion in the electron-ion mass ratio.

THE FOKKER-PLANCK EQUATION

The two equations (\( i = + \) and \( - \)) can be written in the form

\[
\frac{e^i}{m^i} E_r \frac{\partial j^i}{\partial \xi_r} + \frac{\partial}{\partial \xi_r} (\frac{1}{2} b_{ri} j^i) - \frac{\partial}{\partial \xi_r} (a_{ri} j^i) = J^i.
\]

Rationalized mks units are used; \( \kappa \) is the permittivity of free space; \( \lambda \) is the ratio of the Debye length to the mean distance of closest approach in a Coulomb encounter; \( n \) is the ion density. Other convenient forms for the collision term are

\[
J^i = \frac{\partial}{\partial \xi_r} (\frac{1}{2} b_{ri} j^i) - Q^i f^i
\]

(8)

and

\[
J^i = \frac{\partial}{\partial \xi_r} (\frac{1}{2} b_{ri} j^i) + P_r \frac{\partial j^i}{\partial \xi_r} - \frac{\partial Q^i}{\partial \xi_r} j^i.
\]

where

\[
P_r = L_r \sum_j [(m_j - m_i)/m_j] A_{rj}
\]

(10)

and

\[
Q^i = L_r \sum_j (m_j/m_j) A_{rj}.
\]

(11)

The coefficients \( A \) and \( B \) can be evaluated explicitly for a Maxwellian \( f \),

\[
f^0 = n(2\pi RT)^{-\frac{3}{2}} \text{exp} \left( - \frac{1}{2} \xi^2 / 2RT \right).
\]

The summation convention is used for tensor subscripts; time and space variation have been dropped; \( j^i(\xi) \) is the number density in velocity space; \( e^i = \pm e \), the electronic charge. The dynamical friction coefficient \( a_{ri} \) and dispersion coefficient \( b_{ri} \) are given by

\[
a^i(\xi) = L^i \sum_j [(m_j + m_i)/m_j] A_{rj}(\xi)
\]

(2)

and

\[
b^i(\xi) = L^i \sum_j B_{rj}(\xi)
\]

(3)

where

\[
L^i = (e^i / 4\pi \varepsilon^2 m^i) \ln \lambda
\]

(4)

\[
\lambda = 12\pi (\varepsilon kT_i) / (2n_i e^i)
\]

(5)

\[
A_{rj}(\xi) = \int (V_i / V_j) f(\eta) d\eta
\]

(6)

\[
B_{rj}(\xi) = \int [(V^3 d_r - V_r V_d) / V^3] f(\eta) d\eta (V_r = \eta_r - \xi_r).
\]

(7)

\* Institute of Mathematical Sciences, New York University.
we have
\[ A_1^{(0)} = -\frac{1}{2}x^{1/3} + \ldots \]
\[ B_1 = 2x^{1/3} + \ldots \]
\[ B_2 = \frac{3}{2}x^{1/3} + \ldots \]
and for large \( x \),
\[ A_1^{(0)} \sim -x^{1/3} + \ldots \]
\[ B_1 \sim 2x^{1/3} + \ldots \]
\[ B_2 \sim x^{1/3} + \ldots \]
the terms neglected are exponentially small in this case.

The coefficients \( a_r \) and \( b_r \) have a simple intuitive interpretation. A particle which is known to have the velocity \( \xi_r \) at time zero “diffuses” in the interval \( dt \) into a normal distribution peaked about \( \xi_r + a_r dt \) with second moments \( b_{rr} dt \). The equilibrium distribution \( f^{(0)} \) is the result of a competition between a directed frictional force, \( a_r \), tending to bring each particle to rest and a random dispersive “force” \( b_{rr} \) directed outward. In equilibrium, the logarithmic derivative of \( f \) (which is \( -\xi_r/RT \)) is given by the ratio of \( Q \) to \( b_r \) (\( b_r a_r \) are defined as in Eq. (17)).

Many interesting properties can be deduced by inspection of the formulas for \( a_r \) and \( b_r \). For example, the ion friction coefficient \( a^+ \) has a local minimum for a speed somewhere between the mean ion speed \( C_+ \) and the mean electron speed \( C_- \); it is dominated by ion-ion collisions for low ion velocities and by ion-electron collisions for high. Although the transverse component of the dispersion, \( b^- \), is ion dominated for all velocities, the radial component \( b^+ \) becomes electron dominated above a certain speed between \( C_+ \) and \( C_- \). One conclusion we draw is that the relaxation time for that part of the ion distribution which lies between \( C_+ \) and \( C_- \) is in the order of the mean ion-electron relaxation time, which is about \((m_+/m_-)^{1/2} \) slower than the relaxation time for the bulk of the ion distribution.

**LINEARIZATION OF THE EQUATIONS**

For the main part of the distribution, a solution of the form
\[ f(\xi) = f^{(0)}(\xi) + \mathbf{E} \cdot \xi \phi(\xi^2/2) \]
is valid. Let us assume, tentatively, that such a solution has been obtained for both the ion and the electron distributions. Since \( A_r(\xi) \) and \( B_r(\xi) \) are given as integrals over \( f \), they can be computed from (20) and the results are valid for **all values** of \( \xi \). It is assumed (later verified) that the tail of the distribution is exponentially small even though it may differ markedly from \( f^{(0)} \). If we insert the known functions \( A_r(\xi) \) and \( B_r(\xi) \) into the Fokker-Planck equation, the collision term becomes a linear second-order differential operator on \( f \). The complete Fokker-Planck equation is now linear (one need not tamper with the term \( \mathbf{E} \cdot \xi \phi(\xi^2) \), and it is uniformly valid for all values of \( \xi \). Note that the equation is homogeneous; \( \mathbf{E} \) does not multiply an inhomogeneous term, it is a parameter in the differential equation. Because of this we can no longer expect the solution to be linear in \( \mathbf{E} \).

We shall find that the deviation of the ion distribution from Maxwellian is extremely small except for the tail. Consequently, we can take the Maxwellian values \( A^{(0)}, B^{(0)} \) for the ion contributions to \( a^+ \) and \( b^+ \). For the electron coefficients we write
\[ A_r = A_r^{(0)} + A_r^{(1)} \]
\[ B_r = B_r^{(0)} + B_r^{(1)} \]
where
\[ A_r^{(1)} = E_b \int (V_r/V^2) \eta_b \phi(\xi^2) d\eta \]
\[ B_r^{(1)} = E_b \int [(V^2 \delta_{rr} - V_r V_\xi)/V^2 \eta_b] \phi(\xi^2) d\eta. \]

We are interested only in that part of the ion distribution for which \( \xi \ll C_- \). For the electron coefficients (22) we use an expansion about small \( \xi \), obtaining
\[ A_r^{(1)} = (4/15) \pi \phi_1 E_r + \ldots \]
\[ B_r^{(1)} = (8/15) \pi \phi_1 (E_r \xi + E_e \xi_r) \]
where
\[ \phi_1 = \int_0^\infty \phi(x) dx. \]

The value of \( \phi_1 \) can be found without solving for the entire electron distribution function \( \phi(\xi^2) \) by inte-
grating the ion Fokker-Planck equation after multiplying through by $\xi$. The result is

$$\int [a_i^+(\xi) + (e/m_+)E_r] (\xi) d\xi = 0. \quad (25)$$

Using the identity

$$\int A_i^i(\xi)/\xi^2 (\xi) d\xi = 0 \quad (26)$$

for $A^+$ and the approximation (23) for $A^-$, we find

$$\phi_i = -3e\sigma_{m_+}^3/4\pi L^+m_+^2 = 3e\sigma_{m_+}^3/8 \ln \lambda. \quad (27)$$

Surprisingly, the only fact that we need to know about the perturbed electron distribution in order to compute the ion distribution is the single parameter $\phi_i$, which represents the mean frictional force between ions and electrons. This comes out of the formal analysis of the next section, but it can also be explained qualitatively.

The very slight deviation from Maxwellian of the ion distribution is easily explained. To a first approximation, the ion distribution can be considered to be a $\delta$ function on the scale of electron velocities. The electric field accelerates the ion stream relative to the electrons until the mean frictional force balances the applied field. However, the individual ions all see essentially the same value of electron friction; consequently, for each ion the electric field is balanced by the electron friction and, to this order, there is no de-Maxwellizing effect. To the next order of approximation, we consider the electron friction as a linear function over the range of ion velocities instead of a constant value of electron velocities. The approximation (19) and for the electron coefficients the low-velocity approximation (18) and (23). First we transform to spherical coordinates, taking $E$ as the polar axis, $E \cdot \xi = E \cos \theta$, $\xi^2 = r^2$. The coefficients $b_i, \rho_i, q_i$ are functions of $r$ alone. To the order of approximation taken,

$$b_i = (nL^+/C_+)(2/3 + 8/3)(e/m)E^3$$

we obtain

$$\frac{1}{2}b_i \partial^2 \phi/\partial \varphi^2 - b_i E \sin \theta \partial^2 \phi/\partial \varphi \partial \theta + (b_i/2r^e) \partial^2 \phi/\partial \varphi^2 + [b_i r^e + rp_1 + E \cos \theta (p_2 - e/m)] \partial \phi/\partial r$$

$$+ \partial^2 \phi/\theta + E \sin \theta \left(b_i - b_2 + e/m\right)(1/r) \partial \phi/\partial \theta$$

$$- \partial^2 \phi/\partial \varphi \partial \theta + E \cos \theta (4r^2 + r^3 \partial^2 \phi/\partial \varphi)$$

$$\partial q_i/\partial \theta = 0, \quad (30)$$

where we have set

$$b_i = b_i \xi_{\xi} + \xi \xi_{\xi} - \xi \xi_{\xi} \xi_{\xi},$$

$$P_r = \xi + \xi dE_r/\xi$$

The coefficients $b_i, \rho_i, q_i$ are functions of $r$ alone. To the order of approximation taken,

$$b_i = (nL^+/C_+)(2/3 + 8/3)(e/m)E^3$$

we obtain

$$\frac{1}{2}b_i \partial^2 \phi/\partial \varphi^2 - b_i E \sin \theta \partial^2 \phi/\partial \varphi \partial \theta + (b_i/2r^e) \partial^2 \phi/\partial \varphi^2 + [b_i r^e + rp_1 + E \cos \theta (p_2 - e/m)] \partial \phi/\partial r$$

$$+ \partial^2 \phi/\partial \varphi \partial \theta + E \sin \theta \left(b_i - b_2 + e/m\right)(1/r) \partial \phi/\partial \theta$$

$$- \partial^2 \phi/\partial \varphi \partial \theta + E \cos \theta (4r^2 + r^3 \partial^2 \phi/\partial \varphi)$$

$$\partial q_i/\partial \theta = 0, \quad (30)$$

where

$$x = \xi/C_+.$$
theory is adequate. The exponent in \( g \) has order \( \Delta/\epsilon^2 \), say \( 1/\epsilon^2 \) if \( \Delta \sim \epsilon \), and this can be large, producing an enormous perturbation of the Maxwellian distribution. On the other hand, the exponent in the Maxwellian has order \( 1/\epsilon^4 \) for \( x \sim 1/\epsilon^2 \), so the distribution function / itself is very small in this range.

One of the most interesting features of the formula (37) is the minus sign in the exponent. This implies that the ion distribution function is depressed below the Maxwellian in the direction of \( E \) and is enhanced in the opposite direction. Of course, since we have already observed that to lowest order, the forcing term \( E \) is exactly cancelled by the increase in electron friction and it is no longer a matter of simple intuition which direction this higher-order effect will take. As it turns out, the fact is that the electron dispersion coefficient decreases in the direction of \( E \) (the ion distribution function has been shifted in the same direction as \( E \) from the maximum of \( \Theta \)), that produces the effect.

**THERMONUCLEAR OUTPUT**

Since we have computed only the dominant term, i.e., the exponent, of the distribution function, we carry out the thermonuclear computation to the same order. If we write

\[
\sigma \sim \exp \left( \frac{1}{2} mV^2 \right), \quad m = m^+.
\]

(38)

for the D-D cross-section (\( V \) is the relative velocity as in Eq. (6)), the thermonuclear output is given by

\[
\nu \sim \int \exp \left\{ \mu \left( \frac{1}{2} mV^2 \right) \right\} f^{(0)}(\xi) g(\xi) d\xi d\xi.
\]

(39)

With the exception of \( \mu \), the integrand is the exponential of a polynomial in \( \xi \) and \( \xi \). We introduce the variables \( V = \xi - \xi \), \( W = \xi + \xi \) and find that the integrand has a maximum at \( W^{(0)} \) and \( V^{(0)} \) given by

\[
W^{(0)} = -\frac{1}{2} \epsilon^2 (V^{(0)})^2 \epsilon^2/C^+,
\]

\[
\mu'(\frac{1}{2} m(V^{(0)})^2) = (1/2kT)(1 - \frac{1}{2} \epsilon^2 (V^{(0)})^2 \epsilon^2/C^+^2); \quad (40)
\]

\( V^{(0)} \) is the maximizing value for \( \epsilon = 0 \). We approximate the integral by the maximum value of the integrand to obtain

\[
\nu = \nu^{(0)} \exp \left\{ \frac{1}{2} \epsilon^2 (V^{(0)})^2 \epsilon^2/C^+^2 \right\}; \quad (41)
\]

\( \nu^{(0)} \) is the Maxwell-averaged thermonuclear output with \( \epsilon = 0 \). It is convenient to convert \( \theta = kT \) into kev, \( n \) into ions/cm\(^3\) and \( E \) into v/cm. For temperatures which are not too high, we have

\[
m(V^{(0)})^2/\kappa T \sim 12.4/\theta^2
\]

(42)

from which we obtain

\[
\nu = \nu^{(0)} \exp \left\{ (0.9 \times 10^{14} E \theta^4) / n \ln \lambda \right\}
\]

(43)

\[ W^{(0)}/C^+ = 1.5 \times 10^{14} E \theta^4 / n \ln \lambda. \]

(44)

A representative set of values for which the exponent in (43) has order unity might be \( n \sim 10^{14} \text{ ions/cm}^3, E \sim 1 \text{ v/cm, and } \theta \sim 0.5 \text{ kev; } \ln \lambda \) is about 14. The most interesting point about the formula for \( \nu \) is the extremely sensitive dependence on \( n, E \) and \( \theta \). The transition from a negligible correction to an enormous one requires a small change in the parameters. A specific feature is the emphasis on low density; combined with the factor \( n^2 \) which is in \( \nu^{(0)} \), we find that \( \nu \) has a minimum as a function of \( n \) when the exponent is unity and then \( \nu \) rises very rapidly. For example, in a discharge of varying density it is likely that \( E \) would be approximately uniform and the thermonuclear output might then be concentrated in the lower density regions. However, one should recall the fact that the appropriate relaxation time—i.e., the time required to build up the tail of the distribution—is, for one thing, long (comparable to the ion-electron relaxation time) and, for another, inversely proportional to the density.

Formula (44) is also of interest since \( W^{(0)} \) measures the mean shift in the center of mass of the reacting deuterons. The energy spectrum of the neutrons emitted has a shift of the order of the thermal energy when the dimensionless term has order unity. In the example above the shift is about 0.5 kev. We recall that this shift should be observed in the direction opposite to \( E \).

**REFERENCES**