Cyclotron Radiation from a Magnetized Plasma

By S. Hayakawa,* N. Hokkyō,† Y. Terashima* and T. Tsuneto*

For the investigation of physical properties of a plasma and for the interpretation of solar radio outbursts, an important clue may be provided by the cyclotron radiation emitted by electrons gyrating around a static magnetic field. With regard to this we treat two particular problems concerning the effect of fluctuating electric fields; one is the purely random fluctuation which brings about the resonance width and the other the organized plasma oscillation which may give rise to induced emission. The resonance width is shown to be related to the slowing-down time of an electron and the spectral distribution for the dipole radiation is derived under the approximation in which both the higher moment of the resonance shape and the coherence of radiations from a number of electrons are neglected. The angular distribution of the induced emission is expressed in a closed form that can be readily reduced to the formula neglecting the effect of the plasma oscillation. Since the general formula is too lengthy, we give explicit expressions only at particular directions.

I. INTRODUCTION

It is well known that a high-temperature plasma loses energy by bremsstrahlung, according to the formula calculated with the Born approximation, as

$$\frac{dW_b}{dt} = \frac{16}{3}\left(\frac{8}{\pi}\right)^{\frac{1}{2}} \frac{Z^2e^2}{\hbar c}\left(\frac{kT}{m}\right)^{\frac{1}{2}}\left(\frac{e^2}{mc^2}\right)^2 mc^2 n_e n_i, \tag{1.1}$$

where Ze, m, c, $2\pi\hbar$ and k are the ionic charge, electron mass, light velocity, Planck constant and Boltzmann constant, respectively. Measuring the electron temperature, T, in degrees Kelvin and the electron and ion densities, n_e and n_i respectively, in cm^{-3}, the rate of energy loss is expressed by

$$\frac{dW_b}{dt} \cong 1.6\times10^{-27} Z^2 n_i n_e T^{\frac{1}{2}} \text{ erg sec}^{-1}\text{ cm}^{-3} \tag{1.1'}$$

If there exists a magnetic field in a plasma, the so-called cyclotron radiation is expected to compete with the bremsstrahlung. The energy loss due to this process would be

$$\frac{dW_c}{dt} = \frac{4e^2}{3c^3}\frac{kT}{m}\omega_c^2 n_e$$
$$\cong 5.4\times10^{-25} n_e H_0^2 T \text{ erg sec}^{-1}\text{ cm}^{-3} \tag{1.2}$$

The radiation has a line spectrum, the emission frequencies being the integral multiples of the cyclotron frequency,

$$\nu_c = \omega_c/2\pi = eH_0/2\pi mc \cong 2.8\times10^6 H_0 \text{ sec}^{-1} \tag{1.3}$$

where H_0 is the magnetic field strength in gauss. In most practical cases, however, ω_c is smaller than the plasma frequency $\omega_p = (4\pi e^2 n_e/m)^{\frac{1}{2}}$, so that the cyclotron radiation cannot occur, except for its higher harmonics of weak intensity.

The relative importance of the above two modes may be compared by observing that

$$\frac{dW_b}{dt}\bigg/\frac{dW_c}{dt} = \frac{1}{\pi}\left(\frac{8}{\pi}\right)^{\frac{1}{2}}\frac{Ze^2}{\hbar}\left(\frac{m}{kT}\right)^{\frac{1}{2}}\left(\frac{\omega_p}{\omega_c}\right)^2\frac{n_i}{n_e}$$
$$\cong 0.30 n_i Z^2 H_0^{-2} T^{-\frac{1}{2}}, \tag{1.4}$$

and

$$(\omega_p/\omega_c)^2 = 4\pi mc^2 n_e/H_0^2 \cong 1.0\times10^{-5} n_e H_0^{-2}. \tag{1.5}$$

The comparison indicates that the cyclotron radiation becomes important if the magnetic energy density is comparable to the electron mass energy density. Such a condition seems to exist in active celestial regions, such as in a part of the Crab nebula and in the vicinity of an active sun spot. It is not impossible to provide such a condition in laboratory experiments.

If the cyclotron radiation from a plasma is observable, it is also feasible to observe the cyclotron absorption of electro-magnetic waves by the plasma. The absorption coefficient for the dipole mode is given by

$$\kappa(\nu) = (\omega_p^2/6c)\delta(\nu-\nu_c). \tag{1.6}$$

This indicates absorption at a sharply defined frequency, but actually there is a finite absorption width due to the modulation of the gyrating motion of electrons caused by fluctuating electric fields. The width is closely connected with the slowing-down time of an electron.[1] In Sec. 2 we shall show that the reciprocal width is essentially equal to the slowing-down time for dipole radiation; the relation between them becomes complicated for multipole radiations.

The modulation of the electron motion by fluctuating electric fields gives rise not only to the dissipation which results in the emission width or absorption width, but also to a forced motion of electrons that converts the energy of plasma oscillations to radiation, provided that the fluctuation forms an organized

* Kyoto University, Kyoto, Japan.
† Hitachi Central Institute, Tokyo, Japan.

motion over a long period. This mechanism has been discussed by Field,[2] aiming at the interpretation of solar radio outbursts. In Sec. 3 we shall extend his work to a more general case and give an expression, for the intensity of the radiation, that can be directly compared with that for the spontaneous cyclotron radiation; the latter having also been regarded as a possible mechanism for the solar radio outbursts.[3]

2. COLLISION BROADENING OF CYCLOTRON RADIATION

Since in transport phenomena in a plasma the distant collisions play a more important role than the close collisions, we may expect that the distant collisions would be dominant in bringing about the broadening of the cyclotron radiation of electrons. The width due to the close collisions is estimated, according to the Lorentz theory of collision broadening, to be of the order of $1/t_c$, the collision frequency of the Rutherford scattering between electrons. In what follows, therefore, we shall consider only the cumulative effect of the distant collisions on the spectrum of the cyclotron radiation of electrons in a plasma.

As is usually done in calculating transport coefficients for fully ionized plasma, it would be most convenient for our purpose to treat the problem as a sort of Brownian motion. Thus we start from the equation of motion for individual electrons gyrating under an external magnetic field.

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{mc}[\mathbf{v}\times\mathbf{H}_0]-\frac{e}{m}\mathbf{E}_f(\mathbf{v}), \qquad (2.1)$$

where $\mathbf{H}_0$ is the external field and $-e\mathbf{E}_f(\mathbf{v})$ is the randomly fluctuating force exerted by field particles. The vector potential of radiation emitted by an electron and received at time t is given by

$$\mathbf{A} = -\frac{e}{cR_0T}\int_{-T/2}^{T/2}\mathbf{v}(\tau)\delta(\tau-t+R/c)d\tau, \qquad (2.2)$$

where R_0 and R are the distances from the point of observation to the center of gyration and to the electron respectively, and $T = 2\pi/\omega_0$. We shall restrict our discussions to the dipole radiation, in which case $\mathbf{A}$ will become

$$\mathbf{A} = -\frac{e}{cR_0T}\int_{-T/2}^{T/2}\mathbf{v}(\tau)\sum_{n=-\infty}^{\infty}\exp[in\omega_0(\tau-t+R_0\cdot c)]d\tau.$$

The expression for the total intensity of the dipole radiation turns out to be, on averaging over the period T, which is taken to be sufficiently long

$$I = \frac{2}{3}c^{-3}R_0^2|\dot{\mathbf{A}}|^2$$
$$= \frac{2}{3}\left(\frac{e^2}{c^3}\right)\frac{1}{2\pi}\left(\frac{1}{T}\right)^2\sum_{n=-\infty}^{\infty}n^2\omega_0^2\times$$
$$\int\int_{-T/2}^{T/2}\langle\mathbf{v}(\tau_1)\cdot\mathbf{v}(\tau_2)\rangle_{Av}\exp[in\omega_0(\tau_2-\tau_1)]\,d\tau_1 d\tau_2. \qquad (2.3)$$

Here we have taken the average of the quantity $\mathbf{v}(\tau_1)\cdot\mathbf{v}(\tau_2)$ over the great number of electrons. Thus the problem is reduced to finding this autocorrelation of the velocities of electrons from the equation (2.1), the proper treatment of which would be quite difficult and beyond the scope of the present work. In order to obtain the approximate value for it, we rewrite Eq. (2.1) in the form of the Langevin equation, separating the systematic part from $\mathbf{E}_f$:

$$d\mathbf{v}/dt = -(e/mc)[\mathbf{v}\times\mathbf{H}_0]-\eta\mathbf{v}+\mathbf{F} \qquad (2.4)$$

where $\mathbf{F}$ is supposed to be purely random and the friction coefficient, η, is assumed to be constant. Further we shall take into account only the average behavior represented by η, discarding the effect of diffusion due to $\mathbf{F}$ in the velocity space. Then it is easily shown that

$$\langle\mathbf{v}(\tau_1)\cdot\mathbf{v}(\tau_2)\rangle_{Av} = \langle v^2\rangle_{Av}\cos\omega_c(\tau_2-\tau_1)\,e^{-\eta|\tau_2-\tau_1|}. \qquad (2.5)$$

Needless to say, this is valid only for a limited interval of $\tau_2-\tau_1$. When this is inserted in (2.3) we get, after performing the integral and taking the limit of $T\to\infty$,

$$I = \frac{2}{3}\frac{e^2}{c^3}\langle v^2\rangle_{Av}\frac{2\eta}{\pi}\int_0^\infty\frac{\omega^2(\omega^2+\omega_c^2+\eta^2)d\omega}{(\omega^2-\omega_c^2+\eta^2)+4\eta^2\omega_c^2}. \qquad (2.6)$$

Therefore, the width due to the collisions is approximately given by η. For electrons moving with the mean velocity, η is equal to $\langle\Delta v''\rangle/\langle v\rangle$, $\langle\Delta v''\rangle$ being the average rate at which moving electrons are slowed down by the distant collisions. Here we use the value of $\langle\Delta v''\rangle$ calculated for a plasma in the absence of a magnetic field, although it must be admitted that for electrons gyrating under a strong magnetic field the exact value is likely to be different from it. Considering only electron-electron collisions we get with the aid of the result obtained by Chandrasekhar

$$\eta \simeq 3.5n_eT^{-3/2}\ln(D/\bar{r}_R), \qquad (2.7)$$

where D and $\bar{r}_R$ are the Debye shielding radius and the Rutherford scattering radius, respectively, for an electron with mean kinetic energy. The width obtained in this way is apparently $t_c/t_s \simeq (4/3)\ln(D/\bar{r}_R)$ times greater than the one due to the close collisions, where $t_s = 1/\eta$ denotes the slowing down time.

As was emphasized before, the above discussions are only of a qualitative nature, so that it might be relevant here to add some remarks about the assumptions we have made, thereby pointing out possible improvements in the theory. In the first place, the radiation emitted from individual electrons has been assumed to be entirely incoherent, the correlation between the motions of electrons resulting from mutual collisions ignored. It seems that, if taken into account, this will make the width narrower. Secondly, we have taken the expression (2.5) for $\langle\mathbf{v}(\tau_1)\cdot\mathbf{v}(\tau_2)\rangle_{Av}$, that is, we have from the start assumed the Lorentzian type of resonance shape, neglecting the higher moments. Also, diffusion in velocity space has not been considered. Finally, an approximate value has been used for η. In order to improve the theory it is necessary to carry out the detailed analysis of binary collisions under a magnetic field and to find the phase shift of the cyclotron radiation during a collision. We may note that this will enable us at the same time to see how the Larmor gyration would affect the spectrum of the bremsstrahlung. This problem is of some interest because both kinds of radiation compete with each other in usual cases.

As a last remark, we add that it would be worthwhile to evaluate in a more satisfactory manner the autocorrelation of electron velocities, which is known to be of importance in the statistical mechanics of irreversible processes.

3. CYCLOTRON RADIATION INDUCED BY PLASMA OSCILLATIONS

In Eq. (2.1), $\mathbf{E}(t)$ in the right hand side may be a space charge wave propagating with frequency ω and wave vector $\mathbf{k}_L$:

$$\mathbf{E} = \mathbf{E}_L \exp [i(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \tag{3.1}$$

If the angle between $\mathbf{k}_L$ or $\mathbf{E}_L$ and $\mathbf{H}_0$ is θ_L, ω obeys the dispersion relation

$$1 = \left(\omega_p^2 + \frac{3kT}{m}k_L^2\right)\left(\frac{\sin^2\theta_L}{\omega^2-\omega_c^2} + \frac{\cos^2\theta_L}{\omega^2}\right), \tag{3.2}$$

which is an extension of the result obtained by Newcomb[5] for zero temperature.

The electric field (3.1) causes a distorted helical motion of an electron which in turn radiates electromagnetic waves. The nth Fourier component of the vector potential of the radiation field is given by

$$\mathbf{A}_n = (e\omega/2\pi cR_0)e^{ikR}\oint e^{i(n\omega t-\mathbf{k}\cdot\mathbf{r})}d\mathbf{r}, \tag{3.3}$$

where the integral goes over a gyration period; $\mathbf{k}$ is the wave vector of the emitted radiation and makes an angle θ with respect to $\mathbf{H}_0$.

The expression for the radiation intensity in a general direction is so complicated that here we give only the intensities in three special directions. We choose the z axis along $\mathbf{H}_0$ and the x axis parallel to $\mathbf{H}_0\times\mathbf{E}_L$. The azimuthal angle, ϕ, of $\mathbf{k}$ is measured from the x axis.

(1) For $\phi = 0$, the angular distribution for the nth harmonic is expressed as

$$\frac{dI_n}{d\Omega} = \frac{n^2e^2}{2\pi c^3}\left(\frac{eE_L}{m}\right)^2\left\{\frac{\omega^2\omega_c^2}{(\omega^2-\omega_c^2)^2}\sin^2\theta_L|J_n'(x)|^2+\left(\frac{\omega^2}{\omega^2-\omega_c^2}\sin\theta_L\cos\theta-\cos\theta_L\sin\theta\right)^2\frac{n^2}{x^2}|J_n(x)|^2\right\} \tag{3.4}$$

with

$$x = \frac{neE_L}{mc\omega}\left(\frac{\omega^2}{\omega^2-\omega_c^2}\sin\theta_L\sin\theta+\cos\theta_L\cos\theta\right), \tag{3.4'}$$

(2) For $\phi = \pi/2$, we have

$$\frac{dI_n}{d\Omega} = \frac{n^2e^2}{2\pi c^3}\left(\frac{eE_L}{m}\right)^2\left\{\left[\frac{\omega^4}{(\omega^2-\omega_c^2)^2}\sin^2\theta_L\cos^2\alpha+\left(\cos\theta_L\sin\theta\cos\alpha+\frac{\omega\omega_c}{\omega^2-\omega_c^2}\sin\theta_L\cos\theta\sin\alpha\right)^2\right]|J_n'(y)|^2\right.$$
$$\left.+\left[\frac{\omega^4}{(\omega^2-\omega_c^2)}\sin^2\theta_L\sin^2\alpha+(\cos\theta_L\sin\theta\sin\alpha-\frac{\omega\omega_c}{\omega^2-\omega_c^2}\sin\theta_L\cos\theta\cos\alpha\right)^2\right]\frac{n^2}{y^2}|J_n(y)|^2\right\}, \tag{3.5}$$

with

$$y = \frac{neE_L}{mc\omega}\left(\cos^2\theta_L\cos^2\theta+\frac{\omega^2\omega_c^2}{(\omega^2-\omega_c^2)^2}\sin^2\theta_L\sin^2\theta\right)^{\frac{1}{2}}, \quad \alpha = \tan^{-1}\left(\frac{\omega^2-\omega_c^2}{\omega\omega_c}\cot\theta_L\cot\theta\right). \tag{3.5'}$$

(3) For $\theta = \pi/2$, we have

$$\frac{dI_n}{d\Omega} = \frac{n^2e^2}{2\pi c^3}\left(\frac{eE_L}{m}\right)^2\left\{\left[\cos^2\theta_L\cos^2\beta+\frac{\omega^2}{(\omega^2-\omega_c^2)^2}\sin{}^2\theta_L(\omega_c\cos\phi\sin\beta+\omega\sin\phi\cos\beta)^2\right]|J_n'(z)|^2\right.$$
$$\left.+\left[\cos^2\theta_L\sin^2\beta+\frac{\omega^2}{(\omega^2-\omega_c^2)^2}\sin^2\theta_L(\omega_c\cos\phi\cos\beta-\omega\sin\phi\sin\beta)^2\right]\frac{z^2}{n^2}|J_n(z)|^2\right\}, \tag{3.6}$$

with

$$z = \frac{neE_L\sin\theta}{mc\omega}\frac{\omega}{\omega^2-\omega_c^2}(\omega^2\cos^2\phi+\omega_c^2\sin^2\phi)^{\frac{1}{2}}, \quad \beta = \tan^{-1}\{(\omega/\omega_c)\cot\phi\}. \tag{3.6'}$$

In this case, however, there occur complicated stop bands, which should be taken into account when our results are compared with observations.

ACKNOWLEDGEMENTS

Our thanks are due to Professor S. Nakajima and Dr. M. Yokota for their critical but helpful discussions.

REFERENCES

1. L. Spitzer, *Physics of Fully Ionized Gases*, Interscience Publishers, Inc., New York (1956).
2. G. B. Field, Astrophys. J., *124*, 555 (1956).
3. U. E. Kruse, L. Marshall and J. R. Platt, Astrophys. J., *124*, 601 (1956).
4. D. Iwanenko and A. Sokolow, *Classical Field Theory*, Moscow-Leningrad (1949).
5. W. A. Newcomb, NYO–7887 (1954).