Bayesian inference of the properties of plasma turbulence from reflectometry measurements

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Reflectometry is one of the few diagnostic able to probe the electron density fluctuations in tokamak core plasmas

**Problem**

- Due to the absence of a simple relation between the measured signal and the electron density fluctuations, their statistical properties are often considered the same.

**Open questions**

- Is it possible to obtain reliable information on the “true” properties of the turbulence from reflectometer measurements?

*(Gusakov, PPCF, 2011; Blanco, PPCF, 2013)*
Reflectometry is an inverse problem

Bayesian probability theory is the natural framework to solve inverse problems.  
*(Tarantola, Inverse Problem Theory and Methods for Model Parameter Estimation, SIAM, 2005)*

**Goal:** investigate numerically if the properties of the turbulent fluctuations can be properly inferred with Bayesian methods.

- Parameterization of the properties of the turbulent fluctuations.
- Definition of a model for the plasma wave-interaction.
- Application of the Bayes’ theorem to obtain the probability distribution of the turbulence parameters.
The plasma-wave interaction is described with the physical optics approximation \textit{Conway, RSI, 1993}

- A Gaussian beam of width $w$ and wavelength $\lambda$ illuminates the cut-off layer.
- The cut-off layer is modelled as a corrugated surface of length $2L >> w$. The displacement w.r.t. the equilibrium position is $r(x,t)$.
- For normal incidence and reflection (conventional reflectometry), the reflectometer signal is given by:

\[
\rho(t) = A \int_{-L}^{L} e^{-x^2/w^2} e^{i2k r(x,t)} \, dx
\]

- Simulations are performed with a time resolution $\Delta t = 1.5 \times 10^5/f$ (mimics a reflectometer operating at $f=50$ Ghz with 3 $\mu$s time resolution).
- The space and time quantities are normalized to $\lambda$ and $\Delta t$, respectively.
The turbulence is modelled by a time-dependent Gaussian random surface.

The corrugated surface is parameterized by three quantities:

- the **amplitude** of the fluctuations $h$
- the **characteristic size** of the fluctuations $L_c$
- the **characteristic time scale** of the fluctuations $t_c$

The surface of length $2L$ and duration $T$ is obtained in two steps:

- generation of bivariate normal random numbers with mean zero and standard deviation $h$.
- FFT filtering in order to obtain the desired correlation properties.
Bayes’ theorem allows us to compute the probability distribution of the turbulent parameters

Bayes’ theorem: \[ p(A|B)p(B) = p(B|A)p(A) \]

Replacing “A” by “parameters” and “B” by “data”, we see that Bayes’ theorem solves the inverse problem as it allows us to estimate the probability of a given set of parameters given the outcome of an experiment:

\[
p(h, t_c, L_c | \{d_i\}) = \frac{p(\{d_i\} | h, t_c, L_c) p(h, t_c, L_c)}{p(\{d_i\})}
\]

- **Prior**: state of knowledge about the parameters before analysing the data, it can be used to bound the solution:
  - \(-\rho_i < L_c < a\); with \(\rho_i\) and \(a\) the ion gyro radius and minor radius, respectively.
  - \(-1/\omega_{ci} < t_c < \tau_e\); with \(\omega_{ci}\) and \(\tau_e\) the ion cyclotron frequency and the confinement time, respectively.

- **Likelihood**: misfit function between the measured and the simulated reflectometer signal, which takes into account the uncertainties affecting the signal.
The likelihood is estimated with statistical quantities derived from the reflectometer signals

The likelihood is taken as a Gaussian distribution:

\[ \mathcal{L} = \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left( -\frac{(d_i - F_i(h, t_c, L_c))^2}{2\sigma_i^2} \right) \]

**d**: measured signal  \( \sigma \): uncertainties affecting the measured and simulated signals.

**F**: simulated signal obtained with the parameters \( h, t_c, L_c \).

**Which signal should we use?**

Reflectometer signals (phase, power, etc…) are stochastic: can not be used to infer the parameters!

Alternatively, we use **statistical quantities** are they should converge toward fix values.

Two distinct statistical quantities computed on the complex signal are considered:
- the **power spectrum** as it mainly depends on \( t_c \).
- the **histogram** as it mainly depends on \( h \) and \( L_c \).
Finite signal length leads to random uncertainties on the statistics

Histogram

- Suppose that \( C_j \) measurements lie in the \( j^{th} \) bin.
- If \( C_j \) is large enough (\( > 10 \)), then the uncertainty is approximately \( \sigma_j \sim C_j^{1/2} \).

Spectrum

- Suppose that the power spectral density \( S(F) \) is obtained by dividing the raw signal in \( N \) window.
- Then, the uncertainty affecting the power spectral density at the frequency \( F_i \) is approximately \( \sigma_i \sim S(F_i) / N^{1/2} \).
Samples of the posterior distribution $p(h,L_c,t_c|\{d_i\})$ are obtained with a MCMC algorithm.

The Markov Chain Monte Carlo (MCMC) algorithm randomly explores the parameters space by following three rules:

1) Start with an initial guess $\theta_i=\{h,L_c,t_c\}$. Compute the initial likelihood $L(\theta_i)$.

2) Jump randomly in the parameter space from $\theta_i$ to $\theta_j$. Compute the new likelihood $L(\theta_j)$.

3) Accept the new sample $\theta_j$ with probability:
   - $p=1$ if $L(\theta_j) > L(\theta_i)$.
   - $p=L(\theta_j)/L(\theta_i)$ otherwise.
The turbulent parameters are correctly inferred for conditions relevant to tokamak edge plasmas.

<table>
<thead>
<tr>
<th>True value</th>
<th>corr. values in the edge region</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h/\lambda = 0.1$</td>
<td>$\delta n/n \sim 5%$</td>
</tr>
<tr>
<td>$L_c/\lambda = 2$</td>
<td>$\kappa_\theta \rho_s \sim 3$</td>
</tr>
<tr>
<td>$t_c/\Delta t = 10$</td>
<td>$F \sim 30$ kHz</td>
</tr>
<tr>
<td>$w/\lambda = 10$</td>
<td>$w \sim 5$ cm</td>
</tr>
<tr>
<td>$N_t = 2000$ points</td>
<td>$T = 6$ ms</td>
</tr>
<tr>
<td>$N$ samples = $10^4$</td>
<td>-</td>
</tr>
</tbody>
</table>

Even though the signal is integrated over the cut-off layer, information related to the poloidal structure of the fluctuations can be retrieved!
Instrumental noise is treated as an additional parameter that enters the forward model.

Instrumental noise is modelled by additive Gaussian noise $\eta$:

$$\rho(t) = A \int_{-L}^{L} e^{-x^2/\omega^2} e^{i2kr(x,t)} dx + \eta(SNR)$$

reflectometer sig. physical optics approx. Gaussian noise

The noise level is set by the signal-to-noise ratio SNR.

The noise level SNR can be estimated, since random noise on the reflectometer signal leads to a systematic deformation of the statistical properties.
The parameters are still correctly estimated even for noise level SNR = 10dB.
Conclusion: Bayesian probability theory is a promising framework for processing fluctuating reflectometer signals

Main results.
While considering a simple model (Gaussian turbulence + physical optics)
• we correctly inferred the turbulence parameters and we obtained estimates for their uncertainties.
• we showed that the method is robust against additive Gaussian noise. The level of noise can also be estimated.

Perspectives.
• Incorporate more sophisticated models for the turbulence and the plasma-wave interaction.
• Application to experimental data.